

# REGULARITY OF BOUNDARY DATA IN PERIODIC HOMOGENIZATION OF ELLIPTIC SYSTEMS IN LAYERED MEDIA

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**ABSTRACT.** In this note we study periodic homogenization of Dirichlet problem for divergence type elliptic systems when both the coefficients and the boundary data are oscillating. One of the key difficulties here is the determination of the fixed boundary data corresponding to the limiting (homogenized) problem. This issue has been addressed in recent papers by D. Gérard-Varet and N. Masmoudi [8], and by C. Prange [17], however, not much is known about the regularity of this fixed data. The main objective of this note is to initiate a study of this problem, and to prove several regularity results in this connection.

## 1. INTRODUCTION

For a bounded domain  $D \subset \mathbb{R}^d$  ( $d \geq 2$ ) consider the following problem

$$(1.1) \quad -\nabla \cdot \left( A \left( \frac{\cdot}{\varepsilon} \right) \nabla u \right) (x) = 0, \quad x \in D,$$

with oscillating Dirichlet data

$$(1.2) \quad u(x) = g \left( x, \frac{x}{\varepsilon} \right), \quad x \in \partial D.$$

Here  $\varepsilon > 0$  is a small parameter,  $A(x) = (A_{ij}^{\alpha\beta}(x))$  is  $\mathbb{R}^{N^2 \times d^2}$ -valued function defined on  $\mathbb{R}^d$ , where  $1 \leq \alpha, \beta \leq d$ ,  $1 \leq i, j \leq N$ , and the boundary data  $g(x, y)$  is  $\mathbb{R}^N$ -valued function defined on  $\partial D \times \mathbb{R}^d$ . The action of the operator in (1.1) on a vector-function  $u = (u_1, \dots, u_N)$  is defined as

$$-(\mathcal{L}_\varepsilon u)_i(x) := \left[ \nabla \cdot \left( A \left( \frac{\cdot}{\varepsilon} \right) \nabla u \right) \right]_i(x) = \frac{\partial}{\partial x_\alpha} \left[ A_{ij}^{\alpha\beta} \left( \frac{\cdot}{\varepsilon} \right) \frac{\partial u_j}{\partial x_\beta} \right] (x),$$

where  $1 \leq i \leq N$ . Here and throughout the text, if not stated otherwise, we use the summation convention for repeated indices.

**Assumptions.** Here we collect all assumptions which will be used when studying problem (1.1)-(1.2).

- (A1) (Periodicity) The coefficient tensor  $A$  and the boundary data  $g$  in its second (oscillating) variable are  $\mathbb{Z}^d$ -periodic, that is  $\forall y \in \mathbb{R}^d$ ,  $\forall h \in \mathbb{Z}^d$  and  $\forall x \in \partial D$  one has

$$A(y + h) = A(y), \quad g(x, y + h) = g(x, y).$$

- (A2) (Ellipticity) Coefficients are uniformly elliptic and bounded, that is there exist constants  $\Lambda, \lambda > 0$  such that

$$\lambda \xi_\alpha^i \xi_\alpha^i \leq A_{ij}^{\alpha\beta}(x) \xi_\alpha^i \xi_\beta^j \leq \Lambda \xi_\alpha^i \xi_\alpha^i, \quad \forall x \in \mathbb{R}^d, \quad \forall \xi \in \mathbb{R}^{d \times N}.$$

- (A3) (Smoothness) We suppose that the boundary data  $g$  in both variables, all elements of  $A$ , and the boundary of  $D$  are infinitely smooth.

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- (A4) (Geometry of the domain)  $D$  is a strictly convex domain, i.e. the all principal curvatures of  $\partial D$  are bounded away from zero.
- (A5) (Layered medium structure) We assume that the coefficient tensor  $A$  is independent of some fixed rational direction, i.e. there exists a non zero vector  $\nu_0 \in \mathbb{Z}^d$  such that  $(\nu_0 \cdot \nabla)A(y) = 0$  for all  $y \in \mathbb{T}^d$ .

The last hypothesis (A5) models the so-called *laminates*, i.e. when the media has layered structure. For instance, if  $\nu_0 = (0, \dots, 0, 1) \in \mathbb{R}^d$ , we get a model of laminate with respect to the last coordinate. Although homogenization results concerning laminates have been studied in theory, and have independent interest (see e.g. [16]), here the assumption (A5) is technical and is due to our proof.

For each  $\varepsilon > 0$  let  $u_\varepsilon$  be the solution to problem (1.1)-(1.2). Also, for the family of operators  $\{\mathcal{L}_\varepsilon\}_{\varepsilon>0}$  let  $\mathcal{L}_0$  be the homogenized (effective) operator in a usual sense of the theory of homogenization (see e.g. [4]). The following homogenization result for  $u_\varepsilon$  is due to D. Gérard-Varet, and N. Masmoudi.

**Theorem 1.1.** (see [8] Theorem 1.1) *Under assumptions (A1)-(A4) there exists a fixed boundary data<sup>1</sup>  $g^* \in L^\infty(\partial D)$  such that if  $u_0$  solves*

$$\mathcal{L}_0 u_0(x) = 0, \quad x \in D \quad \text{and} \quad u_0(x) = g^*(x), \quad x \in \partial D,$$

*then*

$$\|u_\varepsilon - u_0\|_{L^2(D)} \leq C_\alpha \varepsilon^\alpha, \quad \forall \alpha \in \left(0, \frac{d-1}{3d+5}\right).$$

A result related to Theorem 1.1 was proved in our recent work [2] in collaboration with H. Shahgholian, and P. Sjölin, by an approach different than that of [8]. Define projections  $P_\gamma^k(x) = x_\gamma(0, \dots, 1, 0, \dots) \in \mathbb{R}^N$  with 1 in the  $k$ -th position, where  $1 \leq \gamma \leq d$  and  $1 \leq k \leq N$ . Also, let  $\mathcal{L}_\varepsilon^*$  be the adjoint operator to  $\mathcal{L}_\varepsilon$ , that is the coefficients of  $\mathcal{L}_\varepsilon^*$  are set as  $(A^*)_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$ . We then have the following result.

**Theorem 1.2.** (see [2] Theorem 1.7) *In the same setting as in Theorem 1.1, assume in addition that  $d \geq 3$  and  $\mathcal{L}_\varepsilon^*(P_\gamma^k) = 0$  in  $D$  for all  $1 \leq k \leq N$ ,  $1 \leq \gamma \leq d$ , and any  $\varepsilon > 0$ . Then there exists a function  $g^*$  infinitely smooth on  $\partial D$ , so that if  $u_\varepsilon$  is the solution to (1.1)-(1.2) and  $u_0$  of that with homogenized operator  $\mathcal{L}_0$  and boundary data  $g^*$  then*

$$\|u_\varepsilon - u_0\|_{L^p(D)} \leq C_p [\varepsilon (\ln(1/\varepsilon))^2]^{1/p},$$

*for any  $1 \leq p < \infty$ . Moreover,  $g^*$  may be represented explicitly in terms of the vector field of normals of  $\partial D$ , boundary data  $g$ , the coefficient tensor  $A$  and coefficients of the operator  $\mathcal{L}_0$ .*

Using the periodicity condition on the coefficients  $A$  one may simplify the condition of Theorem 1.2 on  $P_\gamma^k$ -s. Namely, denote  $v_{k,i}^\gamma(x) := (A_{ki}^{\gamma 1}, \dots, A_{ki}^{\gamma d})(x)$ , for  $x \in \mathbb{R}^d$ ,  $1 \leq k, i \leq N$ ,  $1 \leq \gamma \leq d$ , then it is easy to see that the condition  $\mathcal{L}_\varepsilon^*(P_\gamma^k) \equiv 0$  is equivalent to

$$(1.3) \quad \operatorname{div}(v_{k,i}^\gamma)(x) = 0, \quad x \in \mathbb{R}^d, \quad 1 \leq k, i \leq N, \quad 1 \leq \gamma \leq d.$$

In the case of  $N = 1$  (scalar equations) the last condition means that the rows of the matrix  $A$  considered as vector fields in  $\mathbb{R}^d$  must be divergence free. The result concerning regularity of  $g^*$  contained in Theorem 1.2, although restrictive in terms of the structure of the operator  $\mathcal{L}_\varepsilon$ , shows that in some cases one may have smooth boundary data for the homogenized problem. Looking ahead let us remark here, that among other things we will

<sup>1</sup>This theorem is formulated in [8] with  $g^* \in L^p(\partial D)$  for all finite  $p$ . However [8] contains a proof of the stronger statement  $g^* \in L^\infty(\partial D)$ , which we use in the current formulation (in [8] see Proposition 2.4, and the discussion at the end of page 159).

recover this result for  $g^*$  (see subsection 4.5) by a different method which will show the smoothness of  $g^*$  under conditions of Theorem 1.2 in dimension two as well.

Departing from here, we aim at understanding the regularity of the fixed boundary data  $g^*$  defined by Theorem 1.1. Let us first recall some known facts from [8] concerning  $g^*$ . For a unit vector  $n \in \mathbb{S}^{d-1}$  let  $P_{n^\perp}$  be the operator of orthogonal projection on the hyperplane orthogonal to  $n$ . Fix  $l > 0$  so that  $(d-1)l > 1$  and for  $\kappa > 0$  set

$$(1.4) \quad \mathcal{A}_\kappa = \{n \in \mathbb{S}^{d-1} : |P_{n^\perp}(\xi)| \geq \kappa |\xi|^{-l} \text{ for all } \xi \in \mathbb{Z}^d \setminus \{0\}\}.$$

A vector  $n \in \mathbb{S}^{d-1}$  is called *Diophantine*, if  $n \in \mathcal{A}_\kappa$  for some  $\kappa > 0$ . For  $x \in \partial D$  let  $n(x)$  be the unit inward normal at  $x$ , and define  $\Gamma_\kappa = \{x \in \partial D : n(x) \in \mathcal{A}_\kappa\}$ . One can see from the analysis of [8] that for any  $\kappa > 0$  the restriction of  $g^*$  on  $\Gamma_\kappa$  is Lipschitz continuous with the Lipschitz constant bounded by  $C\kappa^{-2}$ , where the constant  $C = C(A, D, g, d)$ . It is shown in [8] that  $\sigma(\mathbb{S}^{d-1} \setminus \mathcal{A}_\kappa) \leq C\kappa^{d-1}$ , where  $\sigma$  denotes the Lebesgue measure on the unit sphere of  $\mathbb{R}^d$ . Also, it is not hard to see that the complement  $\mathcal{A}_\kappa^c = \mathbb{S}^{d-1} \setminus \mathcal{A}_\kappa$ , while a set of small measure, is everywhere dense and is an open subset of the unit sphere. Next, due to strict convexity of  $D$  and smoothness of  $\partial D$ , we have that the Gauss map of  $\partial D$ , namely  $\partial D \ni x \mapsto n(x) \in \mathbb{S}^{d-1}$  is a diffeomorphism, which implies that the sets  $\Gamma_\kappa$  have similar properties as  $\mathcal{A}_\kappa$ , in particular, the surface measure of  $\Gamma_\kappa$  decays as  $\kappa \rightarrow 0$ , and the complement of each  $\Gamma_\kappa$  is open and dense in  $\partial D$ . We see that as  $\kappa \rightarrow 0$ , the sets  $\Gamma_\kappa$  cover the entire boundary of  $D$  up to measure zero, and hence  $g^*$  is defined almost everywhere on  $\partial D$ . However, since the upper bound for Lipschitz constant of  $g^*$  on  $\Gamma_\kappa$ , which is  $C\kappa^{-2}$ , blows up as  $\kappa \rightarrow 0$ , we can *not* conclude that there exists an extension of  $g^*$  to  $\partial D$  which will be continuous at least at a single point. As we will see here, the behaviour of  $g^*$  is more regular for layered structures.

For a given domain  $D$  with smooth boundary, and  $\tau > 0$  set

$$\partial D_\tau = \{x \in \partial D : n(x) \notin \mathbb{R}\mathbb{Q}^d \text{ and } |n(x) \cdot \nu_0| > \tau\},$$

where  $\nu_0$  is fixed from assumption (A5). We have the following result.

**Theorem 1.3.** (The Regularity Theorem) *Let assumptions (A1)-(A5) be in force, and let  $g^*$  be defined by Theorem 1.1. Then, for any  $\tau > 0$  there exists a constant  $C_\tau = C(A, D, g, d, \tau)$  such that*

$$|g^*(x) - g^*(y)| \leq C_\tau |x - y|, \quad \forall x, y \in \partial D_\tau.$$

**Corollary 1.4.**  *$g^*$  has a unique continuous extension to  $\{x \in \partial D : n(x) \cdot \nu_0 \neq 0\}$ .*

**Proof.** Note that by Theorem 1.1  $g^*$  is defined almost everywhere on  $\partial D$  and we need to extend  $g^*$  on a measure zero set of  $\partial D$ . By Theorem 1.3 for any  $\tau > 0$  the function  $g^*$  is uniformly continuous on  $\partial D_\tau$ , and hence admits a unique continuous extension to  $\{x \in \partial D : |n(x) \cdot \nu_0| > \tau\}$ . The proof follows by taking  $\tau \rightarrow 0$ .  $\square$

In general, without any structural assumptions on the operator, we do not know whether  $g^*$  has an extension to  $\partial D$  which is continuous at least at a single point on the boundary. We discuss the construction of  $g^*$  in Section 2, and then prove Theorem 1.3 in Section 4.

The second problem we deal with in this paper concerns Green's kernels arising in homogenization of boundary layer systems set in halfspaces. In particular we establish some regularity results for Green's kernels with respect to the normal direction of the corresponding halfspace. The results are formulated in Section 3.

**Notation.** We fix some notation and conventions that will be used in the sequel. An integer  $d$  always stands for the dimension of  $\mathbb{R}^d$ , and throughout the paper we have  $d \geq 2$ . By  $N \in \mathbb{N}$  we denote the number of equations in (1.1).

$\mathbb{S}^{d-1}$  is the unit sphere, and  $\mathbb{T}^d$  is the unit torus of  $\mathbb{R}^d$ . By  $\mathbb{R}\mathbb{Q}^d$  we denote the set of all vectors from  $\mathbb{R}^d$  that are scalar multiples of vectors with all entries being rational numbers. We call elements of  $\mathbb{R}\mathbb{Q}^d$  *rational* vectors (directions, if they have length one), and the complement of  $\mathbb{R}\mathbb{Q}^d$  is referred to as *irrational* vectors (correspondingly directions).

For a vector  $n \in \mathbb{S}^{d-1}$  we set  $\Omega_n = \{x \in \mathbb{R}^d : x \cdot n > 0\}$ , where “ $\cdot$ ” is the usual inner product in  $\mathbb{R}^d$ . For  $x \in \mathbb{R}^d$ , if no confusion arises we let  $|x|$  be its Euclidean norm. For  $k \in \mathbb{N}$  we denote by  $M_k(\mathbb{R})$  the set of  $k \times k$  matrices with real entries, and by  $O(k)$  the set of  $k \times k$  orthogonal matrices.

Throughout the text various letters  $c, C, C_1, \dots$  denote absolute constants which may vary from formula to formula. For two quantities  $a$  and  $b$  we write  $a \lesssim b$  if there is an absolute constant  $C$  such that  $a \leq Cb$ . If  $a$  and  $b$  depend on some parameter  $\delta$ , then we may write  $a \lesssim_\delta b$  to point out that the constant in the inequality depends on  $\delta$  and is otherwise absolute.

The word “smooth” always means differentiable of class  $C^\infty$ .

## 2. BOUNDARY LAYER SYSTEMS AND CONSTRUCTION OF HOMOGENIZED DATA $g^*$

For a unit vector  $n \in \mathbb{R}^d$  and scalar  $a \in \mathbb{R}$  set  $\Omega_{n,a} = \{x \in \mathbb{R}^d : x \cdot n > a\}$ , and for a smooth and  $\mathbb{Z}^d$ -periodic vector-function  $v_0$  consider the following problem

$$(2.1) \quad \begin{cases} -\nabla \cdot A(y) \nabla v(y) = 0, & y \in \Omega_{n,a}, \\ v(y) = v_0(y), & y \in \partial\Omega_{n,a}. \end{cases}$$

Problems of the form (2.1) will be referred as *boundary layer systems*. Concerning (2.1) we will need the following result.

**Theorem 2.1.** (see [17] Theorem 1.2) *Assume  $n \notin \mathbb{R}\mathbb{Q}^d$ . Then*

1. *there exists a unique solution  $v \in C^\infty(\overline{\Omega_{n,a}}) \cap L^\infty(\Omega_{n,a})$  of (2.1) such that*

$$\|\nabla v\|_{L^\infty(\{y \cdot n > t\})} \rightarrow 0, \text{ as } t \rightarrow \infty,$$

$$\int_a^\infty \|(n \cdot \nabla)v\|_{L^\infty(\{y \cdot n - t = 0\})}^2 dt < \infty,$$

2. *and a boundary layer tail  $v^\infty \in \mathbb{R}^N$  independent of  $a$  so that*

$$v(y) \rightarrow v^\infty, \text{ as } y \cdot n \rightarrow \infty \text{ where } y \in \Omega_{n,a},$$

*and the convergence is locally uniform with respect to the tangential variables.*

Now, following [8] and [17] we describe the construction of the homogenized boundary data. First, consider the case when boundary data  $g$  in (1.2) can be factored into independent components depending on  $x$  and  $y$ . Namely, assume that there exists a smooth  $v_0$  defined on  $\mathbb{T}^d$  with values in  $M_N(\mathbb{R})$  and some smooth  $g_0$  defined on  $\partial D$  and with values in  $\mathbb{R}^N$  so that  $g(x, y) = v_0(y)g_0(x)$ . Next, take any  $x \in \partial D$  such that  $n(x) \notin \mathbb{R}\mathbb{Q}^d$ , and for  $n(x)$  consider the boundary layer system (2.1) with boundary data  $v_0$ . Then let  $v^\infty(x)$  be the constant field provided by Theorem 2.1. Observe, that we do not need to specify the parameter  $a$  in (2.1), since in view of Theorem 2.1 the boundary layer tail  $v^\infty$

<sup>2</sup>It should be remarked that technically Theorem 2.1 is formulated for the case when the boundary data is an  $N$ -dimensional vector, while here we need an  $N \times N$  matrix. Clearly this is not an issue, since one may treat each column of the matrix separately, as is mentioned e.g. in [8].

is independent of  $a$ . Thus, without loss of generality we may assume that  $a = 0$ . Finally, for  $x \in \partial D$  satisfying  $n(x) \notin \mathbb{R}\mathbb{Q}^d$  set

$$g^*(x) := v^\infty(n(x))g_0(x).$$

As we have discussed above, the Gauss map of  $\partial D$  realizes a diffeomorphism between  $\partial D$  and  $\mathbb{S}^{d-1}$ , hence  $g^*$  is defined almost everywhere on  $\partial D$ . The general case proceeds by approximation. Using periodicity of  $g$  in  $y$  and its smoothness we have the following expansion

$$g(x, y) = \sum_{\xi \in \mathbb{Z}^d} c_\xi(x) e^{2\pi i y \cdot \xi} := \sum_{\xi \in \mathbb{Z}^d} g_\xi(x, y),$$

where the series converge uniformly and absolutely. Here  $g_\xi(x, y)$  is factored since  $c_\xi \in \mathbb{R}^N$  and we may identify the exponential  $e^{2\pi i \xi \cdot y}$  with  $e^{2\pi i \xi \cdot y} I_N$ , where  $I_N \in M_N(\mathbb{R})$  is the identity matrix. We let  $v_\xi^\infty$  be the constant field corresponding to the  $\xi$ -th exponential. Then, it is shown in [8] that the homogenized boundary data is given by

$$(2.2) \quad g^*(x) = \sum_{\xi \in \mathbb{Z}^d} c_\xi(x) v_\xi^\infty(n(x)) := \sum_{\xi \in \mathbb{Z}^d} g_\xi^*(x),$$

where  $x \in \partial D$  and  $n(x) \notin \mathbb{R}\mathbb{Q}^d$ . We refer the reader to Section 4.2 of [8] for the details<sup>3</sup>. What we see from here is the fact that the regularity of  $g^*$  depends on the regularity of  $v^\infty$  with respect to the normal directions. To analyse this dependence we will use a representation formula for  $v^\infty$  computed in [17]. For its introduction we start with some preliminaries.

Recall that  $A^*$  is the coefficient tensor for the adjoint operator, i.e.  $(A^*)_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$ . Next, for all  $1 \leq \gamma \leq d$  we let  $v^{*,\gamma} \in M_N(\mathbb{R})$  be the solution (in the sense of Theorem 2.1) to the following system

$$(2.3) \quad \begin{cases} -\nabla_{\tilde{y}} \cdot A^*(\tilde{y}) \nabla_{\tilde{y}} v^{*,\gamma}(\tilde{y}) = 0, & \tilde{y} \in \Omega_n, \\ v^{*,\gamma}(\tilde{y}) = -\chi^{*,\gamma}(\tilde{y}), & \tilde{y} \in \partial\Omega_n, \end{cases}$$

where  $\chi^{*,\gamma} \in M_N(\mathbb{R})$  is the solution to the following *cell-problem*

$$(2.4) \quad \begin{cases} -\nabla_y \cdot A^*(y) \nabla_y \chi^{*,\gamma}(y) = \partial_{y_\alpha} A^{*,\alpha\gamma}, & y \in \mathbb{T}^d, \\ \int_{\mathbb{T}^d} \chi^{*,\gamma}(y) dy = 0. \end{cases}$$

Next, we will need a certain analogue of the notion of *mean-value* for almost-periodic functions.

**Lemma 2.2.** (see [21] Theorem S.3) *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be almost-periodic. Then there exists a scalar  $\mathcal{M}(f)$  such that for any  $\varphi \in L^1(\mathbb{R}^d)$  one has*

$$\int_{\mathbb{R}^d} \varphi(y) f(\lambda y) dy \rightarrow \mathcal{M}(f) \int_{\mathbb{R}^d} \varphi(y) dy, \quad \text{as } \lambda \rightarrow \infty.$$

<sup>3</sup>In fact [8] only treats Diophantine normals in a sense of (1.4). As we have seen above all points of  $\partial D$  up to measure zero satisfy (1.4) for some parameter  $\kappa > 0$ , and hence (2.2) is defined almost everywhere on  $\partial D$ . The extension of (2.2) to all irrational directions follows from Theorem 2.1.

The following useful formula for  $v^\infty(n)$  defined by Theorem 2.1 is due to C. Prange (see formula (6.4) in [17]). Keeping the notation of Theorem 2.1 and Lemma 2.2 we have

$$(2.5) \quad v^\infty(n) = \int_{\partial\Omega_n} \partial_{y_\alpha} G^0(n, y) d\sigma(y) \times \left[ \mathcal{M}\{A^{\beta\alpha}(y)v_0(y)n_\beta\} + \right. \\ \left. \mathcal{M}\left\{\partial_{y_\beta}(\chi^{*,\alpha})^t(y)A^{\beta\gamma}(y)v_0(y)n_\gamma\right\} + \right. \\ \left. \mathcal{M}\left\{\partial_{y_\beta}(v^{*,\alpha})^t(y)A^{\beta\gamma}(y)v_0(y)n_\gamma\right\} \right].$$

Here  $G^0$  is the Green's kernel corresponding to the homogenized constant coefficient operator  $-\nabla \cdot A^0 \nabla$  in domain  $\Omega_n$ . Also, the averages  $\mathcal{M}\{\cdot\}$  are understood for restrictions of functions on the hyperplane  $\Omega_n$ , that is one may apply Lemma 2.2 after rotating the hyperplane  $\partial\Omega_n$  to  $\mathbb{R}^{d-1} \times \{0\}$ . More precisely, for  $F : \mathbb{R}^d \rightarrow \mathbb{R}^N$  and a vector  $n \in \mathbb{S}^{d-1}$  one takes a matrix  $M \in O(d)$  such that  $Me_d = n$  and applies Lemma 2.2 for a function  $f(z') = F(M(z', 0))$ ,  $z' \in \mathbb{R}^{d-1}$ . We do not enter into details concerning almost-periodic functions, as here our treatment will be self-contained. The interested reader is referred, for example, to [21] for particulars.

We finish this section by two observations. First, we compute the constant  $\mathcal{M}$  for some class of almost-periodic functions, and second, we establish a uniform bound on the constant field of Theorem 2.1 in terms of the corresponding boundary data.

**Lemma 2.3.** *Let  $T$  be a fixed  $d \times d$  matrix with rational coefficients, and assume we are given a function  $f(y) = \sum_{\xi \in \mathbb{Z}^d} c_\xi(f) e^{2\pi i T\xi \cdot y}$ ,  $y \in \mathbb{R}^d$ , where each  $c_\xi \in \mathbb{C}$ , and  $\sum_{\xi \in \mathbb{Z}^d} |c_\xi(f)| < \infty$ . For a unit vector  $n \notin \mathbb{R}\mathbb{Q}^d$  and a matrix  $M \in O(d)$  satisfying  $Me_d = n$ , set  $h(z') = f(M(z', 0))$ , where  $z' \in \mathbb{R}^{d-1}$ . Then*

$$\mathcal{M}(h) = \sum_{\xi: T\xi=0} c_\xi(f).$$

**Proof.** To compute  $\mathcal{M}(h)$  fix some  $\varphi \in C_0^\infty(\mathbb{R}^{d-1})$ , set  $\phi_\xi(z') = T\xi \cdot M(z', 0)$  for  $\xi \in \mathbb{Z}^d$  and consider

$$(2.6) \quad \mathcal{J}_\xi(\lambda) = \int_{\mathbb{R}^{d-1}} \varphi(z') e^{-2\pi i \lambda \phi_\xi(z')} dz', \quad \lambda > 1.$$

The proof will be completed once we show that for each  $\xi$  satisfying  $T\xi \neq 0$  one has  $\mathcal{J}_\xi(\lambda) \rightarrow 0$  as  $\lambda \rightarrow \infty$ . We henceforth assume that  $T\xi \neq 0$ .

It follows from the definition of the matrix  $M$  that  $M = [N|n]$ , where  $N$  is a  $d \times (d-1)$  matrix. We have  $T\xi \cdot M(z', 0) = N^t T\xi \cdot z'$ , and hence  $\nabla' \phi_\xi(z') = N^t T\xi$ , for all  $z' \in \mathbb{R}^{d-1}$  where  $\nabla'$  is the gradient in  $\mathbb{R}^{d-1}$ . But as  $M$  is orthogonal, it preserves the Euclidean length, consequently

$$|T\xi| = |M^t T\xi| = |(N^t T\xi, n \cdot T\xi)| = |(\nabla' \phi_\xi(z'), n \cdot T\xi)|.$$

Therefore, if we assume that  $\nabla' \phi_\xi(z') = 0' \in \mathbb{R}^{d-1}$ , we get  $|n \cdot T\xi| = |T\xi|$ , which, by the equality case in Cauchy-Schwarz inequality infers  $n = T\xi/|T\xi|$ . Since  $T$  has rational entries, it follows that  $T\xi \in \mathbb{R}\mathbb{Q}^d$ , and hence so is  $n$ , contradicting the assumption that  $n$  is not rational. We thus conclude that  $\nabla' \phi_\xi(z') \neq 0'$ . Using this, we invoke integration by parts in (2.6) (cf. “the principle of the non-stationary phase” in [20], p. 341, Prop. 4) and get that  $\lim_{\lambda \rightarrow \infty} \mathcal{J}_\xi(\lambda) = 0$ , for any  $\xi \in \mathbb{Z}^d$  with the property  $T\xi \neq 0$ , which completes the proof of the lemma.  $\square$



For vector-valued functions, in view of the linearity of the averaging operator  $\mathcal{M}$ , and choosing matrix-valued test functions in the proof of Lemma 2.3, we immediately get the following.

**Corollary 2.4.** *For  $k \in \mathbb{N}$  assume  $f = (f_1, \dots, f_k)$  where each component  $f_i$  satisfies Lemma 2.3. Similarly, define  $h = (h_1, \dots, h_k)$ . Then*

$$\mathcal{M}(h) = (\mathcal{M}(h_1), \dots, \mathcal{M}(h_k)).$$

Observe that if  $T$  in Lemma 2.3 is the identity matrix, then  $f$  is  $\mathbb{Z}^d$ -periodic, and  $c_\xi$  is the  $\xi$ -th Fourier coefficient of  $f$ . This observation directly implies the independence of the first two averages involved in the formula (2.5) from the normal  $n \notin \mathbb{RQ}^d$ . Namely, since  $A$ ,  $v_0$ , and  $\chi^{*,\gamma}$  are all  $\mathbb{Z}^d$ -periodic, from Lemma 2.3 and Corollary 2.4 we get

$$(2.7) \quad \mathcal{M}\{A^{\beta\alpha}(y)v_0(y)n_\beta\} = \mathcal{M}\{A^{\beta\alpha}(y)v_0(y)\}n_\beta = c_0(A^{\beta\alpha}v_0)n_\beta,$$

and

$$(2.8) \quad \mathcal{M}\left\{\partial_{y_\beta}(\chi^{*,\alpha})^t(y)A^{\beta\gamma}(y)v_0(y)n_\gamma\right\} = \mathcal{M}\left\{\partial_{y_\beta}(\chi^{*,\alpha})^t(y)A^{\beta\gamma}(y)v_0(y)\right\}n_\gamma = c_0[(\chi^{*,\alpha})^t A^{\beta\gamma}v_0]n_\gamma,$$

where we have  $n \notin \mathbb{RQ}^d$ , and  $c_0(f)$  denotes the 0-th Fourier coefficient of  $\mathbb{Z}^d$ -periodic function  $f$ , i.e. the integral of  $f$  over  $\mathbb{T}^d$ . Note that at this stage we are not able to apply Lemma 2.3 to the last average in (2.5).

We will also need a setting when we apply  $\mathcal{M}$  on a one-parameter family of functions. The next statement follows from Lemma 2.3 in a straightforward manner.

**Corollary 2.5.** *Let  $T$ ,  $n$ , and  $M$  be as in Lemma 2.3, and let  $\mathcal{E}$  be some fixed set of parameters. Suppose for each  $\tau \in \mathcal{E}$  we have a function  $f_\tau(y) = \sum_{\xi \in \mathbb{Z}^d} c_\xi(f_\tau) e^{2\pi i T \xi \cdot y}$ ,  $y \in \mathbb{R}^d$ , where each  $c_\xi(f_\tau) \in \mathbb{C}$ ,  $\sum_{\xi \in \mathbb{Z}^d} |c_\xi(f_\tau)| < \infty$ , and for some absolute constant  $C_0$  and some fixed  $k \geq 1$  we have*

$$|c_\xi(f_\tau) - c_\xi(f_\sigma)| \leq C_0(1 + |\xi|^k)|\tau - \sigma|, \quad \tau, \sigma \in \mathcal{E} \text{ and } \xi \in \mathbb{Z}^d.$$

*Then, for any  $g \in C^\infty(\mathbb{T}^d)$ , setting  $h_\tau(z') = (fg)(M(z', 0))$ , where  $z' \in \mathbb{R}^{d-1}$ , we get*

$$|\mathcal{M}(h_\tau) - \mathcal{M}(h_\sigma)| \leq C_g|\sigma - \tau|, \quad \sigma, \tau \in \mathcal{E}.$$

**Proof.** By  $c_\xi(g)$  denote the  $\xi$ -th Fourier coefficient of  $g$ . Then by Lemma 2.3 we have

$$\mathcal{M}(h_\tau) = \sum_{\xi: T\xi \in \mathbb{Z}^d} c_\xi(f_\tau) c_{-T\xi}(g), \quad \tau \in \mathcal{E}.$$

The proof now follows by writing  $|\mathcal{M}(h_\tau) - \mathcal{M}(h_\sigma)| \leq C_0|\sigma - \tau| \sum_{\xi \in \mathbb{Z}^d} (1 + |\xi|^k) |c_\xi(g)|$ , where convergence of the series is due to the smoothness of  $g$ .  $\square$

Again, generalization to the vector-valued case is trivial. We next proceed to a uniform estimate for the boundary layer tail. The claim of the next lemma follows from the Poisson representation of solutions proved in [17] and a bound for Poisson kernel proved in [8]. Due to the lack of an explicit reference we include the proof here.

**Lemma 2.6.** *Keeping the assumptions and notation of Theorem 2.1, for a unit vector  $n \notin \mathbb{RQ}^d$  and boundary data  $v_0$  let  $v^\infty$  be the corresponding constant field. Then there exists a constant  $C = C(A, d)$  independent of  $n$  and  $v_0$ , such that  $|v^\infty| \leq C\|v_0\|_{L^\infty(\mathbb{T}^d)}$ .*

**Proof.** By [17] Section 3.2, for the solution of (2.1) one has

$$v(y) = \int_{\partial\Omega_n} P(y, \tilde{y}) v_0(\tilde{y}) d\sigma(\tilde{y}), \quad y \in \Omega_n,$$

where  $P$  is the Poisson kernel corresponding to (2.1) and satisfies the following estimate (see [8] Lemma 2.5)

$$(2.9) \quad |P(y, \tilde{y})| \leq C \frac{y \cdot n}{|y - \tilde{y}|^d},$$

for all  $d \geq 2$ ,  $y \in \Omega_n$  and  $\tilde{y} \in \partial\Omega_n$ , and the constant  $C$  depending on the operator and dimension  $d$  only. Using (2.9) one gets

$$|v(y)| \leq C \|v_0\|_{L^\infty(\mathbb{T}^d)} \int_{\tilde{y} \cdot n = 0} \frac{y \cdot n}{|y - \tilde{y}|^d} d\sigma(\tilde{y}).$$

For  $M \in O(d)$  satisfying  $n = M e_d$  make a change of variables in the last integral by  $y = Mz$  and  $\tilde{y} = M\tilde{z}$ . Due to orthogonality of  $M$  we have  $M^t n = e_d$ , hence for any  $y \in \Omega_n$  we get  $y \cdot n = z \cdot M^t n = z \cdot e_d = z_d > 0$  from which it follows that

$$|v(Mz)| \leq C \|v_0\|_{L^\infty(\mathbb{T}^d)} z_d \int_{\tilde{z}_d = 0} \frac{d\sigma(\tilde{z})}{|z - \tilde{z}|^d} = C \frac{\|v_0\|_{L^\infty(\mathbb{T}^d)}}{z_d^{d-1}} \int_{\tilde{z}_d = 0} \frac{d\sigma(\tilde{z})}{\left[1 + \sum_{i=1}^{d-1} \left(\frac{z_i - \tilde{z}_i}{z_d}\right)^2\right]^{d/2}}.$$

Setting  $\tau_i := (z_i - \tilde{z}_i)/z_d$ ,  $i = 1, 2, \dots, d-1$  in the last integral, we obtain

$$|v(Mz)| \leq C \|v_0\|_{L^\infty(\mathbb{T}^d)} \int_{\mathbb{R}^{d-1}} \frac{d\tau}{(1 + |\tau|^2)^{d/2}} \leq C \|v_0\|_{L^\infty(\mathbb{T}^d)},$$

finishing the proof.  $\square$

### 3. REGULARITY OF GREEN'S KERNELS WITH RESPECT TO NORMALS

Here we study two regularity problems concerning Green's kernels involved in formula (2.5). The first one studies regularity of integrated kernels as in (2.5) with respect to normals  $n \in \mathbb{S}^{d-1}$ , while the second problem is concerned with the Green's kernels without integration. Interestingly, the former problem is almost immediate, while the latter involves some delicate analysis.

**3.1. Basic preliminaries and the integrated kernel.** For a coefficient tensor  $A$  and a halfspace  $\Omega \subset \mathbb{R}^d$ , the Green's kernel  $G = G(y, \tilde{y}) \in M_N(\mathbb{R})$  corresponding to the operator  $-\nabla \cdot A(y) \nabla$  in domain  $\Omega$  is a matrix function satisfying the following elliptic system

$$(3.1) \quad \begin{cases} -\nabla_y \cdot A(y) \nabla_y G(y, \tilde{y}) = \delta(y - \tilde{y}) I_N, & y \in \Omega, \\ G(y, \tilde{y}) = 0, & y \in \partial\Omega, \end{cases}$$

for any  $\tilde{y} \in \Omega$ , where  $\delta$  is the Dirac distribution and  $I_N \in M_N(\mathbb{R})$  is the identity matrix. To have a quick reference to this situation, we will say that  $G$  is the Green's kernel for the pair  $(A, \Omega)$ . The existence and uniqueness of Green's kernels for divergence type elliptic systems in halfspaces is proved in [11] Theorem 5.4 for  $d \geq 3$ , and for dimension two in [6] Theorem 2.21. Moreover, if  $A^*$  is the coefficient tensor for the adjoint operator, and  $G^*$  is the corresponding Green's kernel, then one has the following symmetry relation

$$(3.2) \quad G^t(y, \tilde{y}) = G^*(\tilde{y}, y), \quad y, \tilde{y} \in \Omega_n.$$

Let  $B^0$  be a constant coefficient elliptic tensor and  $G^0(z, \tilde{z})$  be the Green's kernel for the pair  $(B^0, \mathbb{R}_+^d)$ . Fix a unit vector  $n \in \mathbb{S}^{d-1}$ , along with a matrix  $M \in O(d)$  satisfying



$Me_d = n$ . Note, that we have no assumption on  $n$  being a rational or an irrational direction. For  $y, \tilde{y} \in \Omega_n$  set  $G^n(y, \tilde{y}) := G^0(M^t y, M^t \tilde{y})$ , we now determine a system of equations satisfied by the matrix  $G^n$ .

Clearly, for any  $y \in \partial\Omega_n$  one has  $M^t y \in \partial\mathbb{R}_+^d$  and hence  $G^n(y, \tilde{y}) = 0$ , so we get a zero boundary condition for  $G^n$  in  $\Omega_n$  for any  $\tilde{y} \in \Omega_n$ . To get the system for  $G^n$ , let us rewrite the system in the definition of the Green's kernel in (3.1). Let  $G^0 = (G_{kj}^0) \in M_N(\mathbb{R})$ , then according to (3.1) for all  $1 \leq i, k \leq N$ , we have

$$(3.3) \quad -\partial_{z_\alpha}(B_{ij}^{0,\alpha\beta}\partial_{z_\beta}G_{kj}^0(z, \tilde{z})) = \delta(z - \tilde{z})\delta_{ik}, \quad z \in \mathbb{R}_+^d,$$

where  $\delta_{ik}$  is the Kronecker delta. For fixed  $1 \leq i, j \leq N$  denote  $B_{ij}^0 := (B_{ij}^{0,\alpha\beta}) \in M_d(\mathbb{R})$ , then with this notation (3.3) transforms to

$$-\nabla_z \cdot B_{ij}^0 \nabla_z G_{kj}^0(z, \tilde{z}) = \delta(z - \tilde{z})\delta_{ik}.$$

Now, fix  $\tilde{y} \in \Omega_n$ , then for any  $1 \leq \alpha \leq d$  we have

$$\partial_{y_\alpha} G_{kj}^n(y, \tilde{y}) = \partial_{z_1} G_{kj}^0(M^t y, M^t \tilde{y})m_{\alpha 1} + \dots + \partial_{z_d} G_{kj}^0(M^t y, M^t \tilde{y})m_{\alpha d},$$

and hence  $\nabla_y G_{kj}^n(y, \tilde{y}) = M \nabla_z G_{kj}^0(M^t y, M^t \tilde{y})$ , from which we obtain

$$(3.4) \quad \nabla_y \cdot B_{ij}^0 \nabla_y G_{kj}^n(y, \tilde{y}) = \nabla_z \cdot M^t B_{ij}^0 M \nabla_z G_{kj}^0(z, \tilde{z}),$$

where  $z = M^t y$  and  $\tilde{z} = M^t \tilde{y}$ . Observe that by non-degeneracy of  $M$  we have  $\delta(z - \tilde{z}) = \delta(M^t(y - \tilde{y})) = \delta(y - \tilde{y})$ , which in combination with (3.4) implies the following.

**Claim 3.1.** *Let  $n \in \mathbb{S}^{d-1}$  be any, and  $M \in O(d)$  be such that  $Me_d = n$ . If  $G^{0,n}(z, \tilde{z})$  is the Green's kernel for the pair  $(M^t B^0 M, \mathbb{R}_+^d)$ , then  $G^n(y, \tilde{y}) := G^{0,n}(M^t y, M^t \tilde{y})$  is the Green's kernel for the pair  $(B^0, \Omega_n)$ , where  $M^t B^0 M$  is understood in accordance with (3.4).*

Let us fix a convention which we will use throughout this section when writing  $M^t B^0 M$  as in Claim 3.1. Note, that unless  $N = 1$  (the case of scalar equations) there is a mismatch in the dimensions of  $M$  and  $B^0$ , thus a direct multiplication in the formula  $M^t B^0 M$  is invalid. So, this formula as a matter of fact is a short-cut notation for coefficients we have on the right-hand side of (3.4), and should be understood in a sense of (3.4). We will use this convention without a further reference.

Now let  $G^n(y, \tilde{y})$  be the Green's kernel for the pair  $(A^0, \Omega_n)$ , where  $A^0$  is the homogenized tensor corresponding to  $A(y)$ . For  $1 \leq \alpha \leq d$ , set

$$(3.5) \quad \mathcal{J}^\alpha(n) = \int_{\partial\Omega_n} \partial_{\tilde{y}_\alpha} G^n(n, \tilde{y}) d\sigma(\tilde{y}),$$

which is precisely the term involved in the formula (2.5). Let us stress that  $\mathcal{J}^\alpha(n)$  is well-defined for any  $n \in \mathbb{S}^{d-1}$  and in this section we will not impose any restriction on  $n$ .

We will first study the regularity of  $\mathcal{J}^\alpha$  as a function from the unit sphere  $\mathbb{S}^{d-1}$  to the space of matrices  $M_N(\mathbb{R})$  which for this purpose is identified with  $\mathbb{R}^{N^2}$  in a usual manner. Let  $G^{0,n}(z, \tilde{z})$  be the Green's kernel for the pair  $(M^t A^0 M, \mathbb{R}_+^d)$ , then by Claim 3.1 and the computations preceding that we have

$$\partial_{\tilde{y}_\alpha} G_{jk}^n(n, \tilde{y}) = \partial_{\tilde{z}_1} G_{jk}^{0,n}(e_d, M^t \tilde{y})m_{\alpha 1} + \dots + \partial_{\tilde{z}_d} G_{jk}^{0,n}(e_d, M^t \tilde{y})m_{\alpha d}.$$

Using this we make a change of variables in (3.5) by the formula  $\tilde{y} = M\tilde{z}$ , where  $\tilde{z} \in \mathbb{R}_+^d$ . As  $G^{0,n}$  has zero boundary conditions with respect to both variables, we get that all tangential derivatives in the last expression are vanishing. Also, since  $Me_d = n$  it follows that  $m_{\alpha d} = n_\alpha$  for any  $1 \leq \alpha \leq d$ . We thus get

$$(3.6) \quad \mathcal{J}^\alpha(n) = n_\alpha \int_{\partial\mathbb{R}_+^d} \partial_{\tilde{z}_d} G^{0,n}(e_d, \tilde{z}) d\sigma(\tilde{z}).$$

The following bound is proved in [8] Lemma 2.5 estimate (2.17),

$$|G^{0,n}(z, \tilde{z})| \leq C \frac{z_d \tilde{z}_d}{|z - \tilde{z}|^d}, \quad z \neq \tilde{z} \text{ in } \mathbb{R}_+^d,$$

where  $C$  is independent of  $n$ . Since  $G^{0,n}(e_d, \cdot)$  is zero on  $\partial\mathbb{R}_+^d$ , from the last estimate it easily follows that  $|\nabla_{\tilde{z}} G^{0,n}(e_d, \tilde{z})| \leq C|e_d - \tilde{z}|^{-d}$ , for all  $\tilde{z} \in \partial\mathbb{R}_+^d$ , and hence the integral in (3.6) is absolutely convergent, and is uniformly bounded with respect to  $n$ .

**Remark 3.2.** *Observe, that while  $\mathcal{I}^\alpha(n)$  is independent of the orthogonal matrix  $M$ , the kernel  $G^{0,n}(z, \tilde{z})$  implicitly depends on  $M$ . At this stage the choice of  $M$  is irrelevant, and for the clarity of notation we do not incorporate it into the notation for  $G^{0,n}$ . However, later on, when studying the regularity of kernels  $G^{0,n}$  with respect to  $n$ ,  $M$  will play a key role. The choice of rotation matrices will be specified in subsection 3.2.*

We finish this subsection with a simple observation.

**Lemma 3.3.** *For any  $1 \leq \alpha \leq d$  each component of the matrix function  $\mathcal{I}^\alpha(n) : \mathbb{S}^{d-1} \rightarrow M_N(\mathbb{R})$  is a smooth real-valued function on  $\mathbb{S}^{d-1}$ .*

**Proof.** Set  $\mathcal{I}(n) = \int_{\partial\mathbb{R}_+^d} \partial_{\tilde{z}_d} G^{0,n}(e_d, \tilde{z}) d\sigma(\tilde{z})$ , clearly it is enough to prove the claim for the matrix-function  $\mathcal{I}(n)$ . In view of Claim 3.1 the coefficient tensor corresponding to  $G^{0,n}$  is  $M^t A^0 M =: B^0$ . Next, referring to [17] p. 358, we know that the Poisson's kernel  $P^{0,n} = (P_{ij}^{0,n})_{i,j=1}^N$  corresponding to  $G^{0,n}$  is defined by

$$P_{ij}^{0,n}(z, \tilde{z}) = -B_{kj}^{0,\alpha\beta} \partial_{\tilde{z}_\alpha} G_{ik}^{0,n}(z, \tilde{z})(e_d)_\beta, \quad z \in \mathbb{R}_+^d, \tilde{z} \in \partial\mathbb{R}_+^d.$$

Since  $G^{0,n}$  has zero boundary conditions in  $\mathbb{R}_+^d$  with respect to both of its variables, we have that all tangential derivatives equal to zero in the last expression, and as  $(e_d)_\beta = \delta_{\beta 1}$ , for all  $1 \leq i, j \leq N$  we obtain

$$(3.7) \quad P_{ij}^{0,n}(z, \tilde{z}) = -B_{kj}^{0,dd} \partial_{\tilde{z}_d} G_{ik}^{0,n}(z, \tilde{z}).$$

From definitions of  $B^0$  and  $M$ , for each fixed  $1 \leq k, j \leq N$  we have

$$B_{kj}^{0,dd} = e_d^t B_{kj} e_d = e_d^t M^t A_{kj}^0 M e_d = (M e_d)^t A_{kj}^0 (M e_d) = n^t A_{kj}^0 n \in \mathbb{R}.$$

Combining this with (3.7), for the  $(i, j)$ -th entry of the matrix  $P^{0,n}$  we get

$$(3.8) \quad P_{ij}^{0,n}(z, \tilde{z}) = -n^t A_{kj}^0 n \partial_{\tilde{z}_d} G_{ik}^{0,n}(z, \tilde{z}).$$

For  $n \in \mathbb{S}^{d-1}$  consider the matrix  $A(n) = (a_{kj}(n))_{k,j=1}^N$ , where we have set  $a_{kj}(n) = -n^t A_{kj}^0 n$ . Now, observe that for column-vector  $v_i = (0, \dots, 1, \dots, 0)^t \in \mathbb{R}^N$  with 1 on the  $i$ -th position, and 0 otherwise, we have  $\int_{\partial\mathbb{R}_+^d} P^{0,n}(e_d, \tilde{z}) v_i d\sigma(\tilde{z}) = v_i$  for all  $n \in \mathbb{S}^{d-1}$ , and any  $1 \leq i \leq N$ . This follows from that fact that the unique smooth solution to Dirichlet problem has Poisson integral representation. From here and (3.8) we get

$$I_N = \int_{\partial\mathbb{R}_+^d} P^{0,n}(e_d, \tilde{z}) d\sigma(\tilde{z}) = \mathcal{I}(n) A(n),$$

where as before  $I_N$  is the  $N \times N$  identity matrix. It follows that the matrix  $A(n)$  is invertible for any  $n \in \mathbb{S}^{d-1}$ , and hence  $\mathcal{I}(n) = (A(n))^{-1}$ . On the other hand all components of  $A(n)$  are obviously smooth functions on  $\mathbb{S}^{d-1}$ , therefore the determinant of  $A(n)$  stays away from 0 by compactness of  $\mathbb{S}^{d-1}$ . We conclude that each component of the inverse  $(A(n))^{-1}$  is  $C^\infty$  on  $\mathbb{S}^{d-1}$ , hence we get the claim for  $\mathcal{I}$  and finish the proof of the lemma.  $\square$

It is interesting to observe, that whereas the integral of  $G^{0,n}$  is easily seen to be smooth with respect to  $n$ , proving a similar result for  $G^{0,n}$  itself is comparatively far more involved.

In the remaining part of the section we establish several regularity estimates for kernels  $G^{0,n}$  with respect to  $n$ , and show in particular, that derivative of the integrals in (3.5) can be taken under the integral sign. We note, that while the rest of the section does not directly contribute towards the proof of the regularity result for  $g^*$ , we think that regularity of  $G^{0,n}$  with respect to  $n$  is an interesting problem on its own right. In particular the ideas involved are completely different from those we use to prove the regularity theorem for  $g^*$ .

**3.2. Smooth rotations and universal null-sets.** Here we will study two key auxiliary statements which are needed for our analysis. First problem concerns the choice of orthogonal matrices  $M$  sending  $e_d$  to  $n \in \mathbb{S}^{d-1}$  and varying smoothly with  $n$ . Second statement deals with the null-sets of non-differentiability of Lipschitz mappings which are parameterized in somewhat regular way.

We start with the first problem. Recall that for each  $n \in \mathbb{S}^{d-1}$  we choose a matrix  $M \in O(d)$  such that  $Me_d = n$ . It is straightforward to see from the definition of  $M$  that it is of the form  $M = [N|n]$ , where  $N$  is  $d \times (d-1)$  matrix with the property that its columns form an orthonormal basis in the tangent space of  $\mathbb{S}^{d-1}$  at the point  $n$ . We thus have that  $M$  is defined modulo group  $O(d-1)$ . When studying regularity of Green's kernels with respect to  $n$  we will need the coefficients of the operator corresponding to  $G^{0,n}$  to vary smoothly with  $n$ . As we have seen above, in Claim 3.1, the coefficient tensor for  $G^{0,n}$  equals  $M^t A^0 M$ , and hence we need  $M$  to depend smoothly on  $n$ . Since the first  $(d-1)$  columns of  $M$  form an orthonormal basis for the tangent space of the unit sphere at the point  $n \in \mathbb{S}^{d-1}$  which is being transformed to  $e_d$  by  $M$ , we observe that what we need in effect is a family of smooth vector fields  $\{\mathbf{v}_1(n), \dots, \mathbf{v}_{d-1}(n)\}_{n \in \mathbb{S}^{d-1}}$  that will form an orthonormal basis in the tangent space of  $\mathbb{S}^{d-1}$  at any point  $n$ . Let us fix, that the existence of such vector fields is equivalent to the existence of orthogonal matrices  $M$  sending  $e_d$  to  $n$  and depending smoothly on  $n$ . The existence of the desired vector fields, which may seem intuitively clear at the first sight in view of so many nice properties of the sphere  $\mathbb{S}^{d-1}$ , is actually false in general<sup>4</sup>. Let us very briefly give some details and background on this matter.

A  $C^\infty$ -manifold  $X$  of dimension  $d \geq 1$  is called *parallelizable* if there exist smooth vector fields  $\{\mathbf{v}_1(x), \dots, \mathbf{v}_d(x)\}_{x \in X}$ , such that at each point  $x \in X$  the  $d$ -tuple  $\{\mathbf{v}_i(x)\}_{i=1}^d$  forms a basis in the tangent space of  $X$  at  $x$ . It is well-known that a manifold is parallelizable if and only if its tangent bundle is trivial. On the other hand the tangent bundle of the sphere  $\mathbb{S}^{d-1}$  is trivial if and only if  $d = 1, 2, 4, 8$ . We refer an interested reader to works by Bott, Kervaire, and Milnor [5], [14] for details and proofs. Notice, that parallelizability does not require the basis to be orthonormal, nonetheless, it follows directly that when  $d \notin \{1, 2, 4, 8\}$  one can not fix a family of orthogonal matrices, such that  $Me_d = n$ , and  $M$  varies smoothly with respect to  $n$  globally on all of  $\mathbb{S}^{d-1}$ . For our objective, however, it is enough to have the existence of these smooth fields locally, in the neighbourhood of each point  $n \in \mathbb{S}^{d-1}$ . In the next lemma we give an elementary, self-contained constructive proof of local existence on  $\mathbb{S}^{d-1}$  of the desired vector fields.

**Lemma 3.4.** (Smooth selection of rotations) *Fix any point  $p = (p_1, \dots, p_d) \in \mathbb{S}^{d-1}$ . Then, there exists an open neighbourhood  $\mathcal{P} \subset \mathbb{S}^{d-1}$  of  $p$ , and an assignment  $n \mapsto M_n$  from  $\mathcal{P}$  into  $O(d)$  such that for each  $n \in \mathcal{P}$  we have  $M_n e_d = n$ , and for each  $1 \leq i, j \leq d$ , the real-valued function  $(M_n)_{ij}$  is  $C^\infty$  on  $\mathcal{P}$ .*

**Proof.** The proof is by induction on dimension  $d$ . We will assume that  $p_1 \neq 0$ , and will fix a neighbourhood  $\mathcal{P}$  of  $p$  on  $\mathbb{S}^{d-1}$  where  $|n_1| > |p_1|/2$ , for all  $n \in \mathcal{P}$ . Otherwise,

<sup>4</sup>For  $\mathbb{R}^d$ , with  $d \geq 3$  odd, the non-existence directly follows from Hairy Ball Theorem, which states that there is no non-vanishing continuous, let alone smooth, tangent vector field on even-dimensional spheres.

if  $p_1 = 0$  we may simply permute the coordinate system so that after the permutation the first coordinate of  $p$  is non zero. Thus there is no loss of generality in assuming that  $p_1 \neq 0$ .

Let  $d \geq 2$  be fixed. To a given  $n = (n_1, \dots, n_d) \in \mathcal{P}$  we wish to assign a  $d \times d$  matrix  $X_d(n)$  of the form

$$(3.9) \quad X_d(n) = \begin{pmatrix} a_1(n) & * & \dots & * & n_1 \\ & \dots & & 1 & n_2 \\ \vdots & * & \ddots & & \vdots \\ a_{d-1}(n) & 1 & & 0 & n_{d-1} \\ 1 & 0 & \dots & 0 & n_d \end{pmatrix}$$

where the opposite main diagonal is identically one except the element on the first row; also, with the exception of the  $(d, d)$ -th element everything below the opposite main diagonal is identically zero, and the rest of the elements above the mentioned diagonal are chosen so that to have the following properties:

- (1)  $X_d(n)e_d = n$ ,
  - (2) all columns of  $X_d(n)$  are pairwise orthogonal to each other,
  - (3) for any  $1 \leq i, j \leq d$ , the real-valued function  $n \mapsto [X_d(n)]_{ij}$  is smooth on  $\mathcal{P}$ ,
- where  $[X_d(n)]_{ij}$  is the  $(i, j)$ -th element of the matrix  $X_d(n)$ .

Start with  $d = 2$ , and let  $n = (n_1, n_2) \in \mathcal{P}$  be any. Consider the matrix  $X_2(n) = \begin{pmatrix} a_1(n) & n_1 \\ 1 & n_2 \end{pmatrix}$ , where we have chosen  $a_1(n) = -n_2/n_1$ . Obviously  $X_2(n)$  is of the form (3.9), and satisfies properties (1)-(3) listed above. Now assume we have this construction for dimension  $d - 1$ , and let us construct for  $d$ . We take  $n = (n_1, \dots, n_d) \in \mathcal{P}$ , and set

$$X_d(n) = \begin{pmatrix} a_1(n) & & & & \\ \vdots & & & & \\ & X_{d-1}(n_1, \dots, n_{d-1}) & & & \\ a_{d-1}(n) & & & & \\ 1 & 0 & \dots & 0 & n_d \end{pmatrix},$$

where the vector field  $A_d(n) := (a_1(n), \dots, a_{d-1}(n))$  will be chosen in a moment. By our construction and inductive hypothesis we have that  $X_d(n)$  is of the form (3.9), whatever the choice of the field  $A_d$  is, and hence in particular, condition (1) above is automatically satisfied. Again in view of the inductive hypothesis and the construction of  $X_d(n)$ , starting from the second one all  $(d - 1)$  columns of  $X_d(n)$  satisfy (2) and (3). It is left to determine the field  $A_d(n)$ . Observe that the first column of  $X_d(n)$  is orthogonal to the rest of  $(d - 1)$  columns if and only if  $A_d(n)$  satisfies

$$(3.10) \quad A_d(n)X_{d-1}(n_1, \dots, n_{d-1}) = (0, \dots, 0, -n_d) \in \mathbb{R}^{d-1},$$

where we have treated  $A_d(n)$  as a row-vector. On one hand for each fixed  $n \in \mathcal{P}$ , (3.10) is a system of linear equations with respect to unknowns  $(a_1, \dots, a_{d-1})$ , and with matrix of coefficients equal to  $X_{d-1}$ . On the other hand, by inductive hypothesis we have that all columns of  $X_{d-1}$  are pairwise orthogonal, moreover, by (3.9) we see that all columns of  $X_{d-1}$  considered as  $(d - 1)$ -dimensional vectors have lengths uniformly bounded away from zero when  $n \in \mathcal{P}$ . This, in particular, shows that the determinant of  $X_{d-1}$ , which in this case will be the product of the lengths of its column-vectors in view of the orthogonality

condition, will stay away from zero uniformly as  $n \in \mathcal{P}$ . We thus conclude that the system (3.10) is uniquely solvable for all  $n \in \mathcal{P}$ , and solutions are smooth functions in  $n$  due to inductive hypothesis applied to  $X_{d-1}$ , and Cramer's rule concerning systems of equations. All properties (1)-(3) are now fulfilled, and inductive step is completed.

It is now left to normalize each column of  $X_d(n)$  to unit length, so that to get an orthogonal matrix. For each  $n \in \mathcal{P}$  we let  $M_n$  be the matrix obtained from  $X_d(n)$  where we divide all elements on the given column of  $X_d$  by the Euclidean length of that column-vector. It is important to observe, that the last column of  $X_d(n)$ , which is the vector  $n$ , is of unit length, thus it will remain unchanged leaving the condition of sending  $e_d$  to  $n$  unaltered. The rest of all other  $d-1$  columns of  $X_d(n)$  have length at least one, hence this normalization will not affect the smoothness of the individual components of the matrix. It now follows that the mapping  $\mathcal{P} \ni n \mapsto M_n \in O(d)$  satisfies all requirements of the lemma.

The proof is complete.  $\square$

For each  $n \in \mathbb{S}^{d-1}$ , let  $A_n^0$  be the coefficient tensor for the kernel  $G^{0,n}(z, \tilde{z})$ . Then, by Claim 3.1 we have  $A_n^0 = M^t A^0 M$ , where  $M \in O(d)$  is a matrix that transforms  $e_d$  to  $n$ , and  $A^0$  is the constant tensor corresponding to the homogenized operator  $\mathcal{L}_0$ . One important corollary that we trivially get from the previous lemma, is the possibility of a choice for the coefficients corresponding to rotated Green's kernels, that vary smoothly with respect to normal directions. More precisely, we have the following.

**Corollary 3.5.** *Let  $p = (p_1, \dots, p_d) \in \mathbb{S}^{d-1}$  be fixed. Then there exists  $\mathcal{P} \subset \mathbb{S}^{d-1}$  an open neighbourhood of the point  $p$ , and an assignment  $\mathcal{P} \ni n \mapsto A_n^0$  with  $A_n^0$  as above, such that for each  $1 \leq \alpha, \beta \leq d$ , and each  $1 \leq i, j \leq N$ , the real-valued function  $(A_n^0)_{ij}^{\alpha\beta}$  is  $C^\infty$  on  $\mathcal{P}$ .*

**Proof.** Fix the neighbourhood  $\mathcal{P}$  as in Lemma 3.4, and for  $n \in \mathcal{P}$  let  $M_n \in O(d)$  be the matrix corresponding to  $n$  defined in Lemma 3.4. For this choice of  $M_n$  we have  $A_n^0 = M_n^t A^0 M_n$ , and the claim follows from the smoothness of the assignment  $n \mapsto M_n$ .  $\square$

We now turn to the second problem, mentioned at the beginning of this subsection, and show that for Lipschitz functions, which depend smoothly on some parameter, differentiability set can be chosen independently of the set of parameters.

**Lemma 3.6.** (Universality of null-sets) *Assume for some positive integers  $p, q, m$  we are given open subsets  $\mathcal{N} \subset \mathbb{R}^p$ ,  $X \subset \mathbb{R}^q$ , and a function  $f : \mathcal{N} \times X \rightarrow \mathbb{R}^m$  such that*

- (1)  *$f(\cdot, x)$  is Lipschitz in  $\mathcal{N}$  for any  $x \in X$ , and Lipschitz constant is bounded as a function of  $x$ ,*
- (2) *for difference quotients*

$$\tau_h^{(k)} f(n, x) := \frac{f(n + h e_k, x) - f(n, x)}{h},$$

*where  $k \in \{1, \dots, p\}$ ,  $h > 0$ ,  $n \in \mathcal{N}$ ,  $x \in X$ , and  $e_k$  is the  $k$ -th vector of the standard basis of  $\mathbb{R}^p$ , we have that  $\tau_h^{(k)} f(n, \cdot)$  is Lipschitz on  $X$ , with Lipschitz constant independent of  $h > 0$ , and  $n \in \mathcal{N}$ .*

*Under conditions (1)-(2) there exists a set  $\mathcal{N}_0 \subset \mathcal{N}$  of measure 0, such that for all  $x \in X$  the function  $f(n, x)$  is differentiable in the variable  $n$  at any  $n \in \mathcal{N} \setminus \mathcal{N}_0$ .*

**Remark 3.7.** *By Rademacher's theorem  $f$  is almost everywhere differentiable in  $n$  for each fixed  $x \in X$ . However, a priori these null-sets of non-differentiability depend on  $x$ , and it is not entirely trivial how they behave with respect to  $x$ . The main purpose of this lemma is to show that given some regularity regarding the dependence on the parameter  $x$ ,*

we may have a single null-set of non-differentiability for all  $x \in X$ . Let us also remark, that we do not think that regularity assumptions we impose on  $f$  with respect to  $x$  are optimal, nonetheless the statement of the lemma is enough for applications we are going to need. Given these, here we do not pursue the conclusion of Lemma 3.6 under weaker assumptions, which is definitely a very interesting problem on its own right.

**Proof of Lemma 3.6.** Without loss of generality we will assume that  $m = 1$ , i.e.  $f$  is real-valued, since otherwise we will work with each component of  $f$  individually. Let  $\mathcal{X} := \{x_k\}_{k=1}^\infty$  be a dense and countable subset of  $X$ . For each  $k \in \mathbb{N}$  denote by  $\mathcal{N}_k$  the subset of  $\mathbb{N}$  where  $f(\cdot, x_k)$  is not differentiable in  $n$ . By Rademacher's theorem we have  $\mu(\mathcal{N}_k) = 0$  for any  $k$ , where  $\mu$  is the Lebesgue measure in  $\mathbb{R}^p$ . Now set  $\mathcal{N}_0 := \bigcup_{k=1}^\infty \mathcal{N}_k$ , clearly  $\mu(\mathcal{N}_0) = 0$ , and let us show that  $\mathcal{N}_0$  satisfies the lemma.

Fix some  $x \in X$ , and choose a subsequence of  $\mathcal{X}$ , still labelled as  $\{x_s\}_{s=1}^\infty$  such that

$$(3.11) \quad |x - x_s| \leq 2^{-s}, \quad s = 1, 2, \dots$$

This choice is non empty in view of the density of  $\mathcal{X}$ . For  $k \in \{1, 2, \dots, p\}$  we denote by  $\partial_k$  the differentiation operator with respect to variable  $n$  in the  $k$ -th direction. Now for  $n \in \mathbb{N} \setminus \mathcal{N}_0$ ,  $x \in X$ , and  $h > 0$  we have

$$(3.12) \quad \begin{aligned} & \left| \tau_h^{(k)} f(n, x) - \partial_k f(n, x_s) \right| \leq \\ & \left| \tau_h^{(k)} f(n, x_s) - \partial_k f(n, x_s) \right| + \left| \tau_h^{(k)} f(n, x) - \tau_h^{(k)} f(n, x_s) \right| \leq \\ & \left| \tau_h^{(k)} f(n, x_s) - \partial_k f(n, x_s) \right| + c_0 2^{-s}, \end{aligned}$$

where we have used condition (2) concerning difference quotients, and estimate (3.11) to get the second term in the last row with some absolute constant  $c_0$ . As  $n \in \mathbb{N} \setminus \mathcal{N}_0$ , we have that  $f(\cdot, x_s)$  is differentiable for any  $s$ , and hence  $|\tau_h^{(k)} f(n, x_s) - \partial_k f(n, x_s)| \rightarrow 0$  as  $h \rightarrow 0$ . Using this and getting back to (3.12) one infers

$$(3.13) \quad \partial_k f(n, x_s) - c_0 2^{-s} \leq \liminf_{h \rightarrow 0} \tau_h^{(k)} f(n, x) \leq \limsup_{h \rightarrow 0} \tau_h^{(k)} f(n, x) \leq \partial_k f(n, x_s) + c_0 2^{-s},$$

for all  $s = 1, 2, \dots$ . It is left to prove that  $\partial_k f(n, x_s)$  has a limit as  $s \rightarrow \infty$ . To see it, we observe that

$$(3.14) \quad |\tau_h^{(k)} f(n, x_i) - \tau_h^{(k)} f(n, x_j)| \leq C |x_i - x_j|,$$

for any  $i, j \in \mathbb{N}$ , with constant  $C$  independent of  $h$ , and  $i, j$  due to condition (2) of the lemma. Since  $n \notin \mathcal{N}_0$  the derivative of  $f(\cdot, x_s)$  at the point  $n$  exists for all  $s = 1, 2, \dots$ . Thus taking limits as  $h \rightarrow 0$  in (3.14) we obtain

$$|\partial_i^{(k)} f(n, x_i) - \partial_j^{(k)} f(n, x_j)| \leq C |x_i - x_j|, \quad i, j = 1, 2, \dots,$$

which manifests that the sequence  $\{\partial_k f(n, x_s)\}_{s=1}^\infty$  has a limit as  $s \rightarrow \infty$ . The latter combined with (3.13) shows that the limit of  $\tau_h^{(k)} f(n, x)$  as  $h \rightarrow 0$  exists. Since  $x$  was arbitrary, the proof of the lemma is now complete.  $\square$

**3.3. Stability estimates.** The aim of this subsection is to study regularity with respect to  $n \in \mathbb{S}^{d-1}$  of Green's kernels  $G^{0,n}$ . We start by recording several well-known estimates in elliptic regularity theory, which will be used here. Since we will be working with constant coefficient operators only, we present the estimates for that case only. Also, the estimates are adjusted for our purpose, and are not necessarily in their optimal form.

For  $x_0 \in \partial \mathbb{R}_+^d$ , and  $r > 0$  denote  $D_r(x_0) := B_r(x_0) \cap \mathbb{R}_+^d$ , where  $B_r(x_0)$  is an open ball of radius  $r$  and center at  $x_0$ . Set  $\Gamma_r(x_0) := B_r(x_0) \cap \partial \mathbb{R}_+^d$ , i.e. the flat boundary of the



half-ball  $D_r(x_0)$ . We also let  $B^0$  be a constant coefficient tensor of an elliptic system in divergence form.

- **(Interior gradient estimates)** Assume  $B_r(x_0) \subset \mathbb{R}_+^d$ ,  $f \in C^\infty(\overline{B_r(x_0)})$ , and  $u \in W^{1,2}(B_r(x_0))$  is a weak solution to the system

$$-\nabla \cdot B^0 \nabla u(z) = f(z) \quad \text{in } B_r(x_0).$$

Then, for any finite  $p > d$  one has

$$(3.15) \quad \|\nabla u\|_{L^\infty(B_{r/2}(x_0))} \lesssim_p r^{-1} \|u\|_{L^\infty(B_r(x_0))} + r^{1-\frac{d}{p}} \|f\|_{L^p(B_r(x_0))}.$$

- **(Boundary gradient estimates)** For  $x_0 \in \mathbb{R}_+^d$ , and  $r > 0$  let  $f \in C^\infty(\overline{D_r(x_0)})$ , and  $u \in W^{1,2}(D_r(x_0))$  be a weak solution to the following problem

$$-\nabla \cdot B^0 \nabla u(z) = f(z) \quad \text{in } D_r(x_0) \quad \text{and} \quad u(z) = 0 \quad \text{on } \Gamma_r(x_0).$$

Then, for any finite  $p > d$  one has

$$(3.16) \quad \|\nabla u\|_{L^\infty(D_{r/2}(x_0))} \lesssim_p r^{-1} \|u\|_{L^\infty(D_r(x_0))} + r^{1-\frac{d}{p}} \|f\|_{L^p(D_r(x_0))}.$$

- **(Boundary Hölder estimates)** For  $x_0 \in \mathbb{R}_+^d$ , and  $r > 0$  let  $f \in C^\infty(\overline{D_r(x_0)})$ , and  $u \in W^{1,2}(D_r(x_0))$  be a weak solution to the following problem

$$-\nabla \cdot B^0 \nabla u(z) = \nabla \cdot F(z) \quad \text{in } D_r(x_0) \quad \text{and} \quad u(z) = 0 \quad \text{on } \Gamma_r(x_0).$$

Then, for any finite  $p > d$  one has

$$(3.17) \quad \|u\|_{C^{0,\gamma}(D_{r/2}(x_0))} \lesssim_p r^{-\gamma} \|u\|_{L^\infty(D_r(x_0))} + \|F\|_{L^p(D_r(x_0))},$$

where  $\gamma = 1 - \frac{d}{p}$ .

These estimates are standard and follow from the regularity theory for elliptic systems with constant coefficients, but are hard to locate explicitly in the literature. We refer the reader to [3], Lemmas 16, 20, and 12 for correspondingly *interior gradient*, *boundary gradient*, and *boundary Hölder* estimates. Constants in the inequalities depend on the operator, and dimension of the space, and are independent of the rest of parameters.

We will also need the following estimates concerning Green's kernels  $G^{0,n}$ . For all  $z, \tilde{z} \in \mathbb{R}_+^d$ , satisfying  $z \neq \tilde{z}$ , and any multi-indices  $m, \tilde{m} \in \mathbb{Z}_+^d$  we have

$$(3.18) \quad |\partial_z^m \partial_{\tilde{z}}^{\tilde{m}} G^{0,n}(z, \tilde{z})| \lesssim \frac{1}{|z - \tilde{z}|^{d-2+|m|+|\tilde{m}|}},$$

$$(3.19) \quad |\nabla_z G^{0,n}(z, \tilde{z})| \lesssim \frac{\tilde{z}_d}{|z - \tilde{z}|^d} + \frac{z_d \tilde{z}_d}{|z - \tilde{z}|^{d+1}}.$$

Here constants are independent of  $n$ , and depend upon ellipticity constants of the tensor  $A^0$  and dimension  $d$ . For the proof of (3.18) see [19] V.4.2, Satz 3, and for (3.19) see [8] estimate (2.20) of Lemma 2.5.

The next proposition contains our main results concerning the behaviour of  $G^{0,n}$  with respect to  $n$ . For the kernel  $G^{0,n}(z, \tilde{z})$  by  $\nabla_i$  we set the gradient operator with respect to the  $i$ -th coordinate, where  $i = 1, 2$ , and similarly we let  $\partial_{i,\alpha}$  be the  $\alpha$ -th component of  $\nabla_i$ , where  $1 \leq \alpha \leq d$ . The proof of the proposition below is based on our Lemmas 3.4, and 3.6, and closely follow the scheme of Lemma 6.3 of [17]. The latter on its turn is based on the methods of [3] and [13].

**Proposition 3.8.** (Stability estimates for the Green's kernels) *For each  $p \in \mathbb{S}^{d-1}$  there exists an open neighbourhood  $\mathcal{P} \subset \mathbb{S}^{d-1}$  and an assignment  $\mathcal{P} \ni n \mapsto M_n \in O(d)$  such that if  $G^{0,n}$  is the Green's kernel for the pair  $(M_n^t A^0 M_n, \mathbb{R}_+^d)$ , then for any  $n, \nu \in \mathcal{P}$  we have the following estimates.*

(a) For any  $z, \tilde{z} \in \mathbb{R}_+^d$  with  $z \neq \tilde{z}$ , and any parameter  $0 \leq \gamma < 1$  one has

$$(3.20) \quad |G^{0,n}(z, \tilde{z}) - G^{0,\nu}(z, \tilde{z})| \lesssim_\gamma |n - \nu| \frac{z_d \tilde{z}_d^\gamma}{|z - \tilde{z}|^{d-1+\gamma}}.$$

(b) For any  $\tilde{z} \in \overline{\mathbb{R}_+^d}$  satisfying  $0 \leq \tilde{z}_d \leq 1/8$  one has

$$|\partial_{\tilde{z}_d} G^{0,n}(e_d, \tilde{z}) - \partial_{\tilde{z}_d} G^{0,\nu}(e_d, \tilde{z})| \lesssim \frac{|n - \nu|}{|e_d - \tilde{z}|^{d-1/2}}.$$

(c) The function  $n \mapsto \partial_{\tilde{z}_d} G^{0,n}(e_d, \tilde{z})$  can be differentiated almost everywhere on  $\mathcal{P}$ , with the  $n$ -null-set independent of  $\tilde{z} \in \partial \mathbb{R}_+^d$ .

The absolute constants in the inequalities depend upon ellipticity constants of the homogenized tensor  $A^0$ , dimension  $d$ , and the neighbourhood  $\mathcal{P}$ .

**Proof.** We fix the neighbourhood  $\mathcal{P}$ , and the mapping  $n \mapsto M_n$  defined on  $\mathcal{P}$  from Lemma 3.4. Next, letting  $A_n^0$  be the coefficient tensor corresponding to  $G^{0,n}$ , in view of Corollary 3.5 we get that the mapping  $n \mapsto A_n^0$  is  $C^\infty$  on  $\mathcal{P}$ . In particular for  $n, \nu \in \mathcal{P}$  we have

$$(3.21) \quad |A_n^0 - A_\nu^0| \leq C|n - \nu|,$$

with absolute constant  $C$ .

We first prove the statement of (a) when  $d \geq 3$ . We will start with the following inequality

$$(3.22) \quad |G^{0,n}(z, \tilde{z}) - G^{0,\nu}(z, \tilde{z})| \lesssim \frac{|n - \nu|}{|z - \tilde{z}|^{d-2}}.$$

When  $d \geq 3$  by [11] Corollary 3.5 for any  $z, \tilde{z} \in \mathbb{R}_+^d$  satisfying  $z \neq \tilde{z}$  one has

$$(3.23) \quad G^{0,n}(z, \tilde{z}) - G^{0,\nu}(z, \tilde{z}) = \int_{\mathbb{R}_+^d} \partial_{2,\alpha} G^{0,n}(z, \tilde{z}) (A_\nu^{0,\alpha\beta} - A_n^{0,\alpha\beta}) \partial_{1,\beta} G^{0,\nu}(\tilde{z}, \tilde{z}) d\tilde{z}.$$

We note that this formula is proved for the entire space  $\mathbb{R}^d$ , however the proof of [11] works for the halfspaces as well<sup>5</sup>. Set  $\omega := (z - \tilde{z})/|z - \tilde{z}| \in \mathbb{S}^{d-1}$ . From (3.23), (3.21) and estimates (3.18) concerning derivatives of the Green's kernel we get

$$\begin{aligned} |G^{0,n}(z, \tilde{z}) - G^{0,\nu}(z, \tilde{z})| &\lesssim |n - \nu| \int_{\mathbb{R}^d} \frac{1}{|z - \hat{z}|^{d-1}} \frac{1}{|\hat{z} - \tilde{z}|^{d-1}} d\hat{z} \lesssim \\ &|n - \nu| \int_{\mathbb{R}^d} \frac{1}{|z - \tilde{z} - \hat{z}|^{d-1}} \frac{1}{|\hat{z}|^{d-1}} d\hat{z} \lesssim \\ &\frac{|n - \nu|}{|z - \tilde{z}|^{d-2}} \int_{\mathbb{R}^d} \frac{1}{|\omega - \tilde{z}|^{d-1}} \frac{1}{|\hat{z}|^{d-1}} d\hat{z} \lesssim \frac{|n - \nu|}{|z - \tilde{z}|^{d-2}}, \end{aligned}$$

which is exactly the bound in (3.22).

Armed with (3.22) we now proceed to the proof of gradient bounds for the difference, namely we now show that

$$(3.24) \quad |\nabla_1 G^{0,n}(z, \tilde{z}) - \nabla_1 G^{0,\nu}(z, \tilde{z})| \lesssim \frac{|n - \nu|}{|z - \tilde{z}|^{d-1}}.$$

<sup>5</sup>This nice and useful formula does not come from “nowhere”, and in fact can be seen as a justification of the following heuristic argument. The difference  $G^{0,n}(\cdot, \tilde{z}) - G^{0,\nu}(\cdot, \tilde{z})$  solves the system

$$-\nabla_1 \cdot A_n^0 \nabla_1 (G^{0,n}(\cdot, \tilde{z}) - G^{0,\nu}(\cdot, \tilde{z})) = \nabla_1 \cdot (A_n^0 - A_\nu^0) \nabla_1 G^{0,\nu}(\cdot, \tilde{z})$$

away from the singularity  $\tilde{z}$ . Then if we were allowed to apply Green's representation formula, and then the divergence theorem, we would have exactly recovered (3.23). In [11] a careful analysis involving averaged Green's kernels gives a rigorous proof of (3.23) for  $\mathbb{R}^d$ .

Set  $r := |z - \tilde{z}| > 0$  and consider two cases.

**Case 1:**  $z_d \geq r/10$ . Under this condition on  $z$  we have  $B_{r/10}(z) \subset \mathbb{R}_+^d$ , and hence the difference  $G^{0,n}(\cdot, \tilde{z}) - G^{0,\nu}(\cdot, \tilde{z})$  solves the system

$$-\nabla_1 \cdot A_n^0 \nabla_1 [G^{0,n}(\cdot, \tilde{z}) - G^{0,\nu}(\cdot, \tilde{z})] = \nabla_1 \cdot [A_n^0 - A_\nu^0] \nabla_1 G^{0,\nu}(\cdot, \tilde{z}) \text{ in } B_{r/10}(z).$$

By interior gradient estimates (3.15) for any finite  $p > d$  we have

$$(3.25) \quad |\nabla_1 G^{0,n}(z, \tilde{z}) - \nabla_1 G^{0,\nu}(z, \tilde{z})| \lesssim_p r^{-1} \|G^{0,n}(\cdot, \tilde{z}) - G^{0,\nu}(\cdot, \tilde{z})\|_{L^\infty(B_{r/10}(z))} + r^{1-\frac{d}{p}} \|\nabla_1 \cdot (A_n^0 - A_\nu^0) \nabla_1 G^{0,\nu}(\cdot, \tilde{z})\|_{L^p(B_{r/10}(z))} := r^{-1}(\text{I}) + r^{1-\frac{d}{p}}(\text{II}).$$

By (3.22) we get

$$(3.26) \quad (\text{I}) \lesssim |n - \nu| \sup_{w \in B_{r/10}(z)} \frac{1}{|w - \tilde{z}|^{d-2}} \lesssim |n - \nu| r^{-(d-2)}.$$

To estimate (II) we use the fact coefficients are constant, hence the first gradient in (II) is applied only to the Green's kernel. Next, using (3.21) and (3.19) we obtain

$$(3.27) \quad (\text{II}) \lesssim |n - \nu| \times \|\nabla_1^2 G^{0,\nu}(\cdot, \tilde{z})\|_{L^p(B_{r/10}(z))} \lesssim |n - \nu| r^{\frac{d}{p}} \|\nabla_1^2 G^{0,\nu}(\cdot, \tilde{z})\|_{L^\infty(B_{r/10}(z))} \lesssim r^{\frac{d}{p}} |n - \nu| \sup_{w \in B_{r/10}(z)} \frac{1}{|w - \tilde{z}|^d} \lesssim |n - \nu| r^{\frac{d}{p}} r^{-d}.$$

Combining (3.26), (3.27) into (3.25) immediately leads to gradient bounds of (3.24) under condition of Case 1.

**Case 2:**  $z_d < r/10$ . Here we will use the same reasoning as in Case 1, but instead with boundary gradient estimates. Fix  $\bar{z} \in \partial \mathbb{R}_+^d$  such that  $z_d = |z - \bar{z}|$ . Then the difference  $G^{0,n}(\cdot, \tilde{z}) - G^{0,\nu}(\cdot, \tilde{z})$  solves the following problem

$$(3.28) \quad \begin{cases} -\nabla_1 \cdot A_n^0 \nabla_1 [G^{0,n}(\cdot, \tilde{z}) - G^{0,\nu}(\cdot, \tilde{z})] = \nabla_1 \cdot [A_n^0 - A_\nu^0] \nabla_1 G^{0,\nu}(\cdot, \tilde{z}), & \text{in } D_{r/2}(\bar{z}), \\ G^{0,n}(\cdot, \tilde{z}) - G^{0,\nu}(\cdot, \tilde{z}) = 0, & \text{on } \Gamma_{r/2}(\bar{z}). \end{cases}$$

By exactly the same way as we did in Case 1 we obtain (3.24). The only difference is that instead of (3.15) we apply boundary gradient estimates (3.16) on (3.28). We skip the details, and fix that (3.24) is now proved for all  $d \geq 3$ .

We are now ready to establish (3.20) for  $d \geq 3$ , which will be done in two-step refinements of (3.22).

**Step 1.** We first prove that for all  $z, \tilde{z} \in \mathbb{R}_+^d$  with  $z \neq \tilde{z}$  one has

$$(3.29) \quad |G^{0,n}(z, \tilde{z}) - G^{0,\nu}(z, \tilde{z})| \lesssim \frac{z_d}{|z - \tilde{z}|^{d-1}} |n - \nu|.$$

As before, set  $r := |z - \tilde{z}|$ . Now observe that if  $z_d \geq r/10$ , then (3.29) trivially follows from (3.22) already proved above. We will thus assume that  $z_d < r/10$ . Here as well, let  $\bar{z} \in \partial \mathbb{R}_+^d$  be the tangential component of  $z$ , that is  $z_d = |z - \bar{z}|$ . Since  $G^{0,n}$  has zero boundary values in  $\mathbb{R}_+^d$  we have

$$\begin{aligned} |G^{0,n}(z, \tilde{z}) - G^{0,\nu}(z, \tilde{z})| &= \\ &|G^{0,n}(z, \tilde{z}) - G^{0,\nu}(z, \tilde{z}) - [G^{0,n}(\bar{z}, \tilde{z}) - G^{0,\nu}(\bar{z}, \tilde{z})]| \lesssim \\ &z_d \|\nabla_1 G^{0,n}(\cdot, \tilde{z}) - \nabla_1 G^{0,\nu}(\cdot, \tilde{z})\|_{L^\infty(D_{r/2}(\bar{z}))} \lesssim z_d \frac{|n - \nu|}{r^{d-1}}, \end{aligned}$$

where we have used the mean-value theorem and estimate (3.24). Thus the first refinement (3.29) is proved.

**Step 2.** Here we prove (3.20) when  $d \geq 3$ . Let  $0 < \gamma < 1$  be fixed. Note that  $\gamma = 0$  is already proved in Step 1, and in fact the techniques we are going to use now, do not apply to  $\gamma = 0$ . Again, (3.20) directly follows from (3.29) if  $\tilde{z}_d \geq r/10$ , thus we will assume the opposite inequality. For the coefficient tensor  $A_n^0$  we let  $A_n^{*,0}$  be the tensor corresponding to the adjoint operator, and by  $G^{*,0,n}$  we denote the Green's kernel for the adjoint operator. Recall, that by symmetry we have  $G^{0,n}(z, \tilde{z}) = G^{*,0,n}(\tilde{z}, z)$ . Fix  $\tilde{z} \in \partial\mathbb{R}_+^d$  so that  $\tilde{z}_d = |\tilde{z} - \tilde{z}|$ , and observe that  $G^{*,0,n}(\cdot, z) - G^{*,0,\nu}(\cdot, z)$  solves a problem of the form (3.28) in  $D_{r/2}(\tilde{z})$  and for adjoint coefficients. Note, that  $(A_n^{*,0})_{ij}^{\alpha\beta} = (A_n^0)_{ji}^{\beta\alpha}$ , hence (3.21) holds for adjoint tensors as well. Using boundary Hölder estimates (3.17) we have

$$\begin{aligned}
 (3.30) \quad |G^{0,n}(z, \tilde{z}) - G^{0,\nu}(z, \tilde{z})| &= |G^{*,0,n}(\tilde{z}, z) - G^{*,0,\nu}(\tilde{z}, z)| = \\
 &= |G^{*,0,n}(\tilde{z}, z) - G^{*,0,\nu}(\tilde{z}, z) - [G^{*,0,n}(\tilde{z}, z) - G^{*,0,\nu}(\tilde{z}, z)]| \lesssim \\
 &\quad \tilde{z}_d^\gamma \|G^{*,0,n}(\cdot, z) - G^{*,0,\nu}(\cdot, z)\|_{C^{0,\gamma}(D_{r/4}(\tilde{z}))} \lesssim_p \\
 &\quad \tilde{z}_d^\gamma \left[ r^{-\gamma} \|G^{*,0,n}(\cdot, z) - G^{*,0,\nu}(\cdot, z)\|_{L^\infty(D_{r/2}(\tilde{z}))} + |n - \nu| \times \|\nabla_1 G^{*,0,n}(\cdot, z)\|_{L^p(D_{r/2}(\tilde{z}))} \right] := \\
 &\quad \tilde{z}_d^\gamma [r^{-\gamma}(\text{I}) + |n - \nu| \times (\text{II})],
 \end{aligned}$$

where  $\gamma = 1 - d/p$ , and  $p > d$  is any finite number. To bound (I) we use the symmetry of Green's kernel to pass to  $G^{0,n}$ , and then applying (3.29) we get

$$(3.31) \quad (\text{I}) \lesssim \sup_{w \in D_{r/2}(\tilde{z})} |G^{0,n}(z, w) - G^{0,\nu}(z, w)| \lesssim |n - \nu| \frac{z_d}{r^{d-1}}.$$

For (II) using (3.19) we obtain

$$|\nabla_1 G^{*,0,n}(w, z)| \lesssim \frac{z_d}{|w - z|^d} + \frac{z_d w_d}{|w - z|^{d+1}},$$

but as  $\tilde{z}_d < r/10$ , for all  $w \in D_{r/2}(\tilde{z})$  we have  $\frac{w_d}{|w - z|} \lesssim 1$ , hence  $|\nabla_1 G^{*,0,n}(w, z)| \lesssim \frac{z_d}{|w - z|^d}$ . From here we get

$$(3.32) \quad \|\nabla_1 G^{*,0,n}(\cdot, z)\|_{L^p(D_{r/2}(\tilde{z}))} \lesssim r^{\frac{d}{p}} \|\nabla_1 G^{*,0,n}(\cdot, z)\|_{L^\infty(D_{r/2}(\tilde{z}))} \lesssim r^{d/p} \frac{z_d}{r^d}.$$

Applying (3.31) and (3.32) in (3.30) and taking into account that  $\gamma = 1 - d/p$  directly implies (3.20).

It is left to consider the case  $d = 2$ . This is handled by now a standard trick due to [3] of adding a “dummy” variable to the plane and reducing matters to three dimensional case. More precisely, if  $\mathcal{L}_n := -\nabla \cdot A_n^0 \nabla$  is the operator corresponding to  $G^{0,n}$  in dimension 2, consider a new elliptic operator

$$(3.33) \quad \tilde{\mathcal{L}}_n := \mathcal{L}_n + \frac{\partial^2}{\partial \theta^2}$$

defined on  $\mathbb{R}_+^2 \times \mathbb{T}^1$ , where the torus  $\mathbb{T}^1$  is identified with  $[0, 1)$  in the obvious way. Clearly (3.33) is elliptic, and if we let  $\tilde{G}^{0,n}(z, \theta; \tilde{z}, \tilde{\theta})$  be the Green's kernel corresponding to  $\tilde{\mathcal{L}}_n$  in  $\mathbb{R}_+^2 \times \mathbb{T}^1$ , then by a change of variables we see that

$$(3.34) \quad G^{0,n}(z, \tilde{z}) = \int_0^1 \tilde{G}^{0,n}(z, 0; \tilde{z}, \tilde{\theta}) d\tilde{\theta},$$

where  $z, \tilde{z} \in \mathbb{R}_+^2$ , and  $z \neq \tilde{z}$ . We are now in three dimensional case, and invoking the proof for the case  $d \geq 3$  for  $\tilde{G}^{0,n}$  we get

$$(3.35) \quad |(\tilde{G}^{0,n} - \tilde{G}^{0,\nu})(z, \theta; \tilde{z}, \tilde{\theta})| \lesssim_\gamma |n - \nu| \frac{z_2 \tilde{z}_2^\gamma}{(|z - \tilde{z}|^2 + |\theta - \tilde{\theta}|^2)^{(2+\gamma)/2}},$$

where we have used the fact that the distance of  $(z, \theta) \in \mathbb{R}_+^2 \times \mathbb{T}^1$  from the boundary of the domain is  $z_2$ . Using (3.35) in (3.34) we obtain

$$\begin{aligned} |G^{0,n}(z, \tilde{z}) - G^{0,\nu}(z, \tilde{z})| &\lesssim_\gamma |n - \nu| z_2 \tilde{z}_2^\gamma \int_0^1 \frac{d\tilde{\theta}}{(|z - \tilde{z}|^2 + \tilde{\theta}^2)^{(2+\gamma)/2}} \lesssim_\gamma \\ &|n - \nu| \frac{z_2 \tilde{z}_2^\gamma}{|z - \tilde{z}|^{1+\gamma}} \int_0^\infty \frac{d\tilde{\theta}}{(1 + \tilde{\theta}^2)^{(2+\gamma)/2}}. \end{aligned}$$

The integral is convergent, and we get the inequality of (a) in dimension two as well.

We next prove (b). Fix  $z_0 \in \partial\mathbb{R}_+^d$ , and set  $G(z) := G^{*,0,n}(z, e_d) - G^{*,0,\nu}(z, e_d)$ . By symmetry property (3.2) we have

$$(3.36) \quad \begin{cases} -\nabla \cdot A_n^{0,*} \nabla G(z) = \nabla \cdot (A_n^{0,*} - A_\nu^{0,*}) \nabla G^{*,0,\nu}(z, e_d), & z \in D_{1/4}(z_0), \\ G(z) = 0, & z \in \Gamma_{1/4}(z_0), \end{cases}$$

By the boundary gradient estimates (3.16) for any  $p > d$  we get

$$\|\nabla G\|_{L^\infty(D_{1/8}(z_0))} \lesssim_p \|G\|_{L^\infty(D_{1/4}(z_0))} + \|\nabla \cdot (A_n^{0,*} - A_\nu^{0,*}) \nabla G^{*,0,\nu}(\cdot, e_d)\|_{L^p(D_{1/4}(z_0))}.$$

Using part (a) with  $\gamma = 1/2$ , the fact that the coefficients are constant, and estimate (3.18) for derivatives of the kernel, from the last inequality we get

$$(3.37) \quad \|\nabla_z G^{*,0,n}(z, e_d) - \nabla_z G^{*,0,\nu}(z, e_d)\|_{L^\infty(D_{1/8}(z_0))} \lesssim \frac{|n - \nu|}{|e_d - z|^{d-1/2}} + \frac{|n - \nu|}{|e_d - z|^d}.$$

By continuity (3.37) holds up to the boundary. Part (b) now follows by combining (3.37) with (3.2).

It is left to establish (c). We will simply check that all conditions of Lemma 3.6 are satisfied for the function  $f(n, \tilde{z}) := \partial_{\tilde{z}_d} G^{0,n}(e_d, \tilde{z})$ , where  $n \in \mathcal{P}$ , and  $\tilde{z} \in \mathbb{R}^{d-1}$ . A small adjustment one has to keep in mind here is that Lemma 3.6 works with open subsets of Euclidean space, while here we have an open subset of the sphere. But as differentiability of functions defined on a sphere is understood in terms of local coordinate charts, things are reduced to open subsets of Euclidean space. To keep the notation more concise we will work on a sphere directly, having in mind that the sphere is locally represented as a graph. With this convention, the first condition of Lemma 3.6 follows from (b). For the second one, concerning difference quotients, observe that if  $\tau_h^{(k)} f(n, \tilde{z})$  is the difference quotient of Lemma 3.6, then by (3.36) we have

$$(3.38) \quad \begin{cases} -\nabla_{\tilde{z}} \cdot A_n^{0,*} \nabla_{\tilde{z}} \tau_h^{(k)} G^{0,n}(n, \tilde{z}) = \nabla_{\tilde{z}} \cdot \left( \tau_h^{(k)} A_n^{0,*} \right) \nabla_{\tilde{z}} G^{*,0,\nu}(z, e_d), & \tilde{z} \in D_{1/4}(z_0), \\ \tau_h^{(k)} G^{0,n}(n, \tilde{z}) = 0, & \tilde{z} \in \Gamma_{1/4}(z_0), \end{cases}$$

The supremum norm of  $\tau_h^{(k)} G^{0,n}(n, \tilde{z})$  is bounded by part (b), and the source term of (3.38) is bounded in  $C^\infty$  by the choice of coefficients and (3.18). Moreover, all these bounds hold uniformly with respect to  $n$  and  $h$ . Now, by standard elliptic regularity we obtain that  $\tau_h^{(k)} f(n, \cdot)$  is Lipschitz with respect to its second variable and Lipschitz constant is independent of  $n$  and  $h$ . Given these, (c) follows from Lemma 3.6.

The proof of the proposition is now complete.  $\square$

**Corollary 3.9.** *Let  $\mathcal{J}^\alpha$  be defined as in (3.5). Then, there is a choice of rotations  $n \mapsto M_n \in O(d)$  where  $M_n e_d = n$  such that the derivative of  $\mathcal{J}^\alpha$  can be taken under the integral sign.*

**Proof.** Choose rotations as in Proposition 3.8. Then, the corollary easily follows by assertions (b) and (c) of the proposition.  $\square$

#### 4. THE REGULARITY OF $g^*$

The objective of this section is to prove Theorem 1.2. Observe, that so far we had no recourse to assumption (A5) regarding the layered structure, and it is here that it will play a central role in the analysis.

**4.1. Change of variables.** At several places in this section we will switch from one variable to another; we record the necessary details here. Let  $y \in \mathbb{R}^d$  and for a coefficient tensor  $B = B^{\alpha\beta}(y) \in M_N(\mathbb{R})$  which is smooth and elliptic in a sense of standard assumptions (A2) and (A3) of Section 1 consider the operator  $\mathcal{L} = -\nabla_y \cdot B(y) \nabla_y$ . For  $x \in \mathbb{R}^d$  set  $y = Tx$ , where  $T \in M_d(\mathbb{R})$  and has non zero determinant. One may easily deduce that

$$(4.1) \quad \nabla_y = (T^t)^{-1} \nabla_x = (T^{-1})^t \nabla_x.$$

For  $1 \leq i, j \leq N$  let  $B_{ij}$  be the  $d \times d$  matrix formed from the  $(i, j)$ -th entries of the matrices  $B^{\alpha\beta}$ . Then using (4.1) we see that the operator  $\mathcal{L}$  in the new variable  $x$  can be written as  $\mathcal{L} = -\nabla_x \cdot \tilde{B}(Tx) \nabla_x$ , where correspondingly

$$(4.2) \quad \tilde{B}_{ij}(Tx) = T^{-1} B_{ij}(Tx) (T^{-1})^t$$

for all  $1 \leq i, j \leq N$ . To keep track of the ellipticity constant of the new operator we take a family of vectors  $\xi = \xi^\alpha \in \mathbb{R}^N$ , set  $\omega_i = (\xi_i^1, \dots, \xi_i^d)^t$  where  $1 \leq i \leq N$  and compute

$$(4.3) \quad \tilde{B}_{ij}^{\alpha\beta} \xi_j^\beta \xi_i^\alpha = \omega_i^t \tilde{B}_{ij} \omega_j = \omega_i^t T^{-1} B_{ij} (T^{-1})^t \omega_j = [(T^{-1})^t \omega_i]^t B_{ij} (T^{-1})^t \omega_j \geq$$

$$\lambda_B \sum_{i=1}^N \|(T^{-1})^t \omega_i\|^2 \geq \lambda_B \sigma_{\min}^2(T^{-1}) \omega_i \cdot \omega_i = \lambda_B \sigma_{\min}^2(T^{-1}) \xi^\alpha \cdot \xi^\alpha,$$

where  $\lambda_B$  is the ellipticity constant of the original operator and  $\sigma_{\min}(T^{-1})$  is the least singular value of the matrix  $T^{-1}$ , that is the square root of the smallest eigenvalue of  $T^{-1}(T^{-1})^t$ . In particular, it follows that the new operator is elliptic, with possibly a different ellipticity constant.

**4.2. Solutions with exponentially decaying gradients.** Here we use assumption (A5) to gain some extra control on solutions to boundary layer systems. To illustrate what one can get from (A5) we will start with a simple example involving the Laplace operator.

**Example 4.1.** Assume  $N = 1$ , i.e we have only one equation, and for an *irrational* direction  $n \in \mathbb{S}^{d-1}$  and  $u_0 \in C^\infty(\mathbb{T}^d)$  consider the following problem

$$(4.4) \quad \Delta u = 0 \text{ in } \Omega_n \quad \text{and} \quad u = u_0 \text{ on } \partial\Omega_n.$$

Let  $\{c_\xi(u_0)\}_{\xi \in \mathbb{Z}^d}$  be the sequence of Fourier coefficients of  $u_0$ . Then, by a direct computation one can easily check that the function

$$(4.5) \quad u(y) = \sum_{\xi \in \mathbb{Z}^d} c_\xi(u_0) e^{-2\pi [|\xi|^2 - (n \cdot \xi)^2]^{\frac{1}{2}} (y \cdot n)} e^{2\pi i \xi \cdot [y - n(y \cdot n)]}, \quad y \in \overline{\Omega_n},$$

solves (4.4) and satisfies all requirements of Theorem 2.1. It follows in particular that  $u$  defined by (4.5) is the unique solution of (4.4) given by Theorem 2.1. Since  $n \notin \mathbb{RQ}^d$ , the



equality case of the Cauchy-Schwarz inequality provides  $|\xi|^2 - (n \cdot \xi)^2 \neq 0$  unless  $\xi = 0$  and hence the boundary layer tail in this case is simply  $c_0(u_0)$ .

Now assume that  $u_0$  is independent of the last coordinate, i.e.  $(e_d \cdot \nabla)u_0 = 0$  on  $\mathbb{T}^d$ . This condition can be reformulated in terms of Fourier coefficients. Namely, using the smoothness of  $u_0$  and applying  $e_d \cdot \nabla$  on the Fourier series of  $u_0$ , by Parseval's identity we obtain that  $\xi_d c_\xi(u_0) = 0$  for all  $\xi \in \mathbb{Z}^d$ . The latter implies that  $c_\xi(u_0) = 0$  for any  $\xi \in \mathbb{Z}^d$  with  $\xi_d \neq 0$ , that is the Fourier spectrum of  $u_0$  is contained in the sublattice  $\mathbb{Z}^{d-1} \times \{0\} \subset \mathbb{Z}^d$ . Next, suppose the unit vector  $n$  satisfies  $n_d \neq 0$ . Then for  $\xi = (\xi', 0) \in \mathbb{Z}^{d-1} \times \{0\}$  we have

$$(n \cdot \xi)^2 \leq (n_1^2 + \dots + n_{d-1}^2)|\xi'|^2 = (1 - n_d^2)|\xi|^2,$$

therefore

$$||\xi|^2 - (n \cdot \xi)^2| = |\xi|^2 \left| 1 - \frac{(n \cdot \xi)^2}{|\xi|^2} \right| \geq n_d^2 |\xi|^2.$$

The latter combined with (4.5) illustrates that given the special structure of the Fourier spectrum of  $u_0$ , the solution of (4.4) converges exponentially fast in the direction of the normal vector  $n$  towards its boundary layer tail. Also, it is clear that the decay properties deteriorate as  $n_d \rightarrow 0$ . It should also be noted that while  $u_0$  was independent of  $e_d$ , the solution  $u$  does not necessarily satisfy this independence criterion.

To treat the general case we will need a construction due to L. Tartar. For  $Y'$ , an open parallelepiped in  $\mathbb{R}^{d-1}$ , set  $G = Y' \times [0, \infty)$ . Let  $f = \{f_i\}$  and  $F = \{F_i^\alpha\}$  be given smooth functions, where  $1 \leq i \leq N$  and  $1 \leq \alpha \leq d$ . For the unknown vector  $u = (u_1, \dots, u_N)$  consider the following problem

$$(4.6) \quad \begin{cases} -\nabla \cdot A(y) \nabla u(y) = f - \nabla \cdot F(y), & \text{in } G, \\ u(y', 0) = 0, & y' \in Y', \\ u(\cdot, y_d), & \text{is } Y'\text{-periodic for any } y_d > 0, \end{cases}$$

where the system of equations is understood as follows

$$-\frac{\partial}{\partial y_\alpha} \left( A_{ij}^{\alpha\beta}(y) \frac{\partial u_j}{\partial y_\beta}(y) \right) = f_i - \frac{\partial F_i^\alpha}{\partial y_\alpha}, \quad i = 1, 2, \dots, N.$$

We assume that there exists  $\tau_0 > 0$  such that

$$(4.7) \quad e^{\tau_0 y_d} f(y) \in L^2(G; \mathbb{R}^N) \quad \text{and} \quad e^{\tau_0 y_d} F(y) \in L^2(G; \mathbb{R}^{d \times N}),$$

and

$$(4.8) \quad f(\cdot, y_d) \text{ and } F(\cdot, y_d) \text{ are both } Y'\text{-periodic for any } y_d \geq 0.$$

Next, we introduce spaces of functions whose gradients decay exponentially. Let  $H_{per}^1(Y')$  be the space of all functions from  $H^1(Y')$  which have equal traces on opposite faces of  $Y'$ . Now for  $\tau > 0$  set

$$V_\tau = \{v : v \in L_{loc}^2(\mathbb{R}_+; H_{per}^1(Y')), e^{\tau y_d} \nabla v(y) \in L^2(G) \text{ and } v(y', 0) \equiv 0\}.$$

One can see that  $V_\tau$  is a Hilbert space with scalar product defined by

$$[u, v]_\tau = \int_G e^{2\tau y_d} \nabla u(y) \cdot \nabla v(y) dy.$$

The norm on  $V_\tau(G)$  induced from the scalar product is denoted by  $\|\cdot\|_{V_\tau(G)}$ . The existence of solutions to (4.6) with exponentially decaying gradients is given in the following result.

**Theorem 4.2.** (see [18] Chapter 18, and [15] Theorem 10.1) *Assume (4.7), (4.8) and that the coefficient tensor in (4.6) is bounded and is uniformly elliptic with ellipticity constant  $\lambda_A > 0$ . Then for any  $0 < \tau < \min\{\tau_0, \frac{\lambda_A}{2\|A\|_\infty}\}$  there exists a unique solution  $u$  to system (4.6) satisfying  $e^{\tau y_d} \nabla u(y) \in L^2(G; \mathbb{R}^{d \times N})$ . More precisely, for any such  $\tau > 0$  one has the following estimate*

$$(4.9) \quad \|u\|_{V_\tau(G)} \leq \frac{C}{\tau} \frac{1}{\lambda_A - 2\tau\|A\|_\infty} [\|e^{\tau y_d} f\|_{L^2(G; \mathbb{R}^N)} + \|e^{\tau y_d} F\|_{L^2(G; \mathbb{R}^{d \times N})}],$$

where the constant  $C$  depends on dimension  $d$  and the parallelepiped  $Y'$ .

Observe, that at this stage we do not use periodicity of  $A$ , nor any other structural restriction is imposed on the operator. The following useful fact follows immediately from the Theorem 4.2.

**Corollary 4.3.** *Let  $g$  be smooth and periodic vector-function defined on  $Y'$  with values in  $\mathbb{R}^N$ . Then the system*

$$(4.10) \quad \begin{cases} -\nabla \cdot A(y) \nabla u(y) = 0, & y \in G = Y' \times [0, \infty), \\ u(y', 0) = g(y'), & y' \in Y', \\ u(\cdot, y_d), & \text{is } Y'\text{-periodic for any } y_d \geq 0, \end{cases}$$

has a unique solution  $u$  such that  $e^{\tau y_d} \nabla u \in L^2(G; \mathbb{R}^{d \times N})$  for any  $0 < \tau < \frac{\lambda_A}{2\|A\|_\infty}$ . Moreover, the solution  $u$  satisfies

$$(4.11) \quad \|u\|_{V_\tau(G)} \leq \frac{C}{\tau} \frac{\|A\|_\infty}{\lambda_A - 2\tau\|A\|_\infty} \|g\|_{H^1(Y'; \mathbb{R}^N)}.$$

**Proof.** Take any non-negative, compactly supported smooth function  $\varphi(y_d)$ , such that  $\varphi = 1$  near  $y_d = 0$ . Set  $\tilde{g}(y) = \varphi(y_d)g(y')$ , for  $y = (y', y_d) \in G$ . Since  $\tilde{g}$  has compact support in the direction of  $e_d$  we have that  $e^{\tau y_d} \nabla \tilde{g} \in L^2(G; \mathbb{R}^{d \times N})$ , for any  $\tau > 0$ . Now let  $\tilde{u}$  be the unique solution to the problem (4.6) with the right-hand side  $\nabla \cdot A(y) \nabla \tilde{g}$ . Defining  $u = \tilde{u} + \tilde{g}$  we get a solution to (4.10). The estimate (4.11) follows easily from the corresponding estimate of Theorem 4.2.  $\square$

**Remark 4.4.** *The formulation of Theorem 4.2 is slightly more general than the original one as given e.g. in [15] or [18]. Namely, here it is stated for elliptic systems rather than scalar equations, and involves detailed estimates of  $V_\tau$  norms of solutions. The proof however, follows the lines of the original proof, making small changes to deal with systems of equations, and keeping track of the norms of the quantities involved in norm estimate of the Theorem.*

**4.3. The case when  $\nu_0 = e_d$ .** We will first carry out the analysis when the vector  $\nu_0$  defined from assumption (A5) coincides with  $e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$ . To fix the ideas, we let  $A = A^{\alpha\beta} \in M_N(\mathbb{R})$  be a coefficient tensor satisfying the standard ellipticity, smoothness, and periodicity conditions introduced in Section 1, and in addition we require  $A$  to be independent of  $e_d$ . We also fix  $v_0 \in C^\infty(\mathbb{T}^d)$  which is assumed to be independent of  $e_d$  as well. Then, for a given  $n \in \mathbb{S}^{d-1}$  consider the following problem

$$(4.12) \quad \begin{cases} -\nabla_y \cdot A(y) \nabla_y v(y) = 0, & y \in \Omega_n, \\ v(y) = v_0(y), & y \in \partial\Omega_n. \end{cases}$$

Let  $v$  be the unique solution to (4.12) given by Theorem 2.1 when  $n \notin \mathbb{RQ}^d$ . The aim now is to show that this solution has some extra regularity properties given the structural restriction on  $A$  and  $v_0$ .

We will assume that  $n_d \neq 0$ , and then without loss of generality will take  $n_d > 0$ , as the case  $n_d < 0$  works in the same way. The case of  $n_d = 0$  is degenerate, and the analysis breaks down. Also, notice that at this stage we do not require  $n$  to be irrational. To the unit vector  $n = (n_1, \dots, n_d)$  we attach a matrix  $T_n \in M_d(\mathbb{R})$  given by

$$(4.13) \quad T_n = \begin{pmatrix} & & 0 \\ & I_{d-1} & \vdots \\ -\frac{n_1}{n_d} & \dots & -\frac{n_{d-1}}{n_d} & 1 \end{pmatrix},$$

where  $I_{d-1} \in M_{d-1}(\mathbb{R})$  is the identity. It is clear that a linear transformation associated with  $T_n$  is a bijection from  $\overline{\mathbb{R}_+^d}$  to  $\overline{\Omega_n}$ , and that

$$(4.14) \quad T_n^{-1} = \begin{pmatrix} & & 0 \\ & I_{d-1} & \vdots \\ \frac{n_1}{n_d} & \dots & \frac{n_{d-1}}{n_d} & 1 \end{pmatrix}$$

is the inverse of  $T_n$ . We make a change of variables in (4.12) by setting  $y = T_n z$ , where  $z \in \mathbb{R}_+^d$ . Following the notation and results of Section 4.1, if we let  $A_n$  be the coefficient tensor in the new variable  $z$  then

$$(4.15) \quad (A_n)_{ij}(T_n z) = T_n^{-1} A_{ij}(T_n z) (T_n^{-1})^t$$

But observe that since  $A$  is assumed to be independent of  $e_d$ , it follows from (4.13) that  $A_n(T_n z) = A(z)$ . Similarly  $v_0(T_n z) = v_0(z)$ , as  $v_0$  is also taken to be independent of  $e_d$ . Thus, the problem (4.12) is being transformed to

$$(4.16) \quad \begin{cases} -\nabla_z \cdot A_n(z) \nabla_z w(z) = 0, & z \in \mathbb{R}_+^d, \\ w(z) = v_0(z), & z \in \partial \mathbb{R}_+^d, \end{cases}$$

where  $A_n$  is given by (4.15) and  $w(z) = v(T_n z)$ . The ellipticity of  $A_n$  follows from non-degeneracy of  $T_n$  and (4.3). We now give an estimate on the ellipticity constant of  $A_n$  which we will use in the sequel. Following (4.3) we need to bound the smallest singular value of  $T_n^{-1}$  from below, which is being done using the following result.

**Theorem 4.5.** (see [10] Theorem 1) *For a matrix  $T \in M_d(\mathbb{C})$  let  $r_i(T)$  be the Euclidean norm of its  $i$ -th row,  $c_i(T)$  be the Euclidean norm of its  $i$ -th column, and set  $r_{\min}(T) = \min_{1 \leq i \leq d} r_i(T)$  and  $c_{\min}(T) = \min_{1 \leq i \leq d} c_i(T)$ . Then, for  $\sigma_{\min}(T)$ , the smallest singular value of  $T$ , one has*

$$(4.17) \quad \sigma_{\min}(T) \geq \left( \frac{d-1}{d} \right)^{(d-1)/2} |\det T| \max \left\{ \frac{c_{\min}(T)}{\prod_{i=1}^d c_i(T)}, \frac{r_{\min}(T)}{\prod_{i=1}^d r_i(T)} \right\}.$$

We have  $\det(T_n^{-1}) = 1$  and using the fact that  $|n| = 1$  we obtain  $r_{\min}(T_n^{-1}) / \prod_{i=1}^d r_i(T_n^{-1}) = n_d$ . Now by virtue of (4.17) it follows that

$$(4.18) \quad \sigma_{\min}(T_n^{-1}) \geq \left( \frac{d-1}{d} \right)^{(d-1)/2} n_d.$$

Hence, for  $\lambda_{A_n}$ , the ellipticity constant of the operator in (4.16), we have by (4.3) and (4.18) that

$$(4.19) \quad \lambda_{A_n} \geq c_d \lambda_A n_d^2,$$

where  $c_d$  is a constant depending on the dimension, and  $\lambda_A$  is the ellipticity constant of the original operator.

By Corollary 4.3 we get that the following problem

$$(4.20) \quad \begin{cases} -\nabla_z \cdot A_n(z) \nabla_z w(z) = 0, & z \in G = Y' \times \mathbb{R}_+, \\ w(z', 0) = v_0(z', 0), & z' \in Y' = (0, 1)^{d-1}. \end{cases}$$

admits a unique solution in the space  $V_\tau(G)$  for each  $0 < \tau < \frac{\lambda_{A_n}}{2\|A_n\|_\infty}$ . Since all the data are smooth in (4.20) it follows from the standard elliptic regularity for weak solutions that  $w \in C^\infty(G)$  (see e.g. [9] Corollary 4.12). By using the periodicity of the operator and the boundary data in (4.20) as well as the fact that the gradient of  $w$  has equal trace on the opposite sides of  $Y'$  we extend  $w$  by periodicity to the entire halfspace  $\mathbb{R}_+^d$ . By this we obtain a smooth solution to (4.16) which we denote by  $w_n$ . Also, as the boundary data is smooth we get that  $w_n$  is smooth up to the boundary of  $\mathbb{R}_+^d$  (see e.g. [9] Theorem 5.21). Moreover, we have by construction and the definition of  $V_\tau$  spaces that  $w(\cdot, z_d)$  is  $\mathbb{Z}^{d-1}$ -periodic for each  $z_d \geq 0$  and has exponentially decaying gradient in the direction of  $e_d$ . As  $w_n$  solves (4.16) it follows that  $v_n(y) = w_n(T_n^{-1}y)$  solves (4.12), where  $y \in \overline{\Omega_n}$ . We now need to check that  $v_n$  coincides with  $v$  which was the solution to (4.12) given by Theorem 2.1. For that we will use the next lemma, which, as well as the initial idea of exploiting layered structure of the problem were motivated by [16].

**Lemma 4.6.** *For  $n \in \mathbb{S}^{d-1}$  satisfying  $n_d > 0$ , let  $v_n$  be the solution to (4.12) constructed as above. Then  $v_n \in C^\infty(\overline{\Omega_n}) \cap L^\infty(\overline{\Omega_n})$  and satisfies the following properties*

- (a)  $\|\nabla v_n\|_{L^\infty(\{y \cdot n > t\})} \rightarrow 0$ , as  $t \rightarrow \infty$ ,
- (b)  $\int_0^\infty \|(n \cdot \nabla) v_n\|_{L^\infty(\{y \cdot n = t\})}^2 dt < \infty$ .

**Proof.** We have  $v_n(y) = w_n(T_n^{-1}y)$  where  $y \in \overline{\Omega_n}$ , hence the up to the boundary smoothness of  $v_n$  directly follows from that of  $w_n$ .

Fix some  $\tau > 0$  so that  $w_n \in V_\tau$  and let  $z \in \mathbb{R}_+^d$ . Since  $w_n$  solves (4.16), where the coefficients and the boundary data have bounded  $C^k$  norms for any  $k \geq 0$ , by standard Schauder estimates near the boundary (see [9] Theorem 5.21) we have that  $|\nabla_z w_n(z)| \leq C$  uniformly for all  $z = (z', z_d) \in \mathbb{R}^{d-1} \times [0, \infty)$  satisfying  $z_d \leq 1$ . We now estimate  $|\nabla_z w_n(z)|$  for  $z_d > 1$ . By  $\mathbb{Z}^{d-1}$ -periodicity of  $w_n(\cdot, z_d)$  we may assume that  $z' \in (0, 1)^{d-1}$ . For  $r > 0$  let  $K(z, r)$  be a closed cube centred at  $z$  and side length  $r$ . In view of interior Schauder estimates (see [9], Theorem 5.19) we have

$$(4.21) \quad \|\nabla w_n\|_{L^\infty(K(z, 1/4))} \leq C \|\nabla w_n\|_{L^2(K(z, 1/2))},$$

with constant  $C$  independent of  $z$ . Set  $K'(z, 1/2)$  to be the  $(d-1)$ -dimensional cube which is the projection of  $K(z, 1/2)$  onto  $\mathbb{R}^{d-1} \times \{0\}$ . We have

$$(4.22) \quad \begin{aligned} \|\nabla w_n\|_{L^2(K(z, 1/2))}^2 &= \int_{K'(z, 1/2)} \int_{z_d-1/2}^{z_d+1/2} |\nabla w_n(x', x_d)|^2 dx' dx_d \leq \\ &e^\tau e^{-2\tau z_d} \int_{K'(z, 1/2)} \int_{z_d-1/2}^{z_d+1/2} |e^{\tau x_d} \nabla w_n(x', x_d)|^2 dx' dx_d \leq e^\tau e^{-2\tau z_d} \|w_n\|_{V_\tau(G)}^2, \end{aligned}$$

where  $G = (0, 1)^{d-1} \times \mathbb{R}_+$ , and we have used the periodicity of  $w$  to get a bound in  $V_\tau(G)$  norm. From (4.22) and (4.21) we obtain

$$(4.23) \quad |\nabla w_n(z)| \leq C e^{-\tau z_d} \|w_n\|_{V_\tau(G)}, \quad z \in \mathbb{R}_+^d,$$

where we have also included the case of  $z_d \leq 1$  in view of the uniform bound on the gradient. Both assertions of the lemma follow directly from (4.23) and the relation  $\nabla_y v(y) = (T_n^{-1})^t \nabla_z w_n(z)$ , with  $y = T_n z$  which is due to the change of variables formula.

Finally, by writing

$$w_n(z) = w_n(z', z_d) = v_0(z', 0) + \int_0^{z_d} \partial_t w_n(z', t) dt$$

and using (4.23) we get  $w \in L^\infty(\overline{\Omega_n})$ , and complete the proof of the lemma.  $\square$

By Lemma 4.6,  $v_n$  gives a smooth and bounded solution to (4.12), and satisfies condition 1 of Theorem 2.1. But the solution with these properties is unique according to Theorem 2.1. Hence we have the following.

**Corollary 4.7.** *Fix  $n \notin \mathbb{R}\mathbb{Q}^d$  such that  $n_d > 0$ , and assume that in (4.12)  $A$  and  $v_0$  are independent of  $e_d$  and satisfy the usual ellipticity, smoothness, and periodicity assumptions of Section 1. Then the solution  $v_n$  of (4.12) coincides with the one given by Theorem 2.1.*

From the properties of  $w_n$  we now deduce an expansion for  $v_n$ . For  $n \in \mathbb{S}^{d-1}$  satisfying  $n_d > 0$  let  $w_n$  be the solution to (4.16) constructed as above. Then due to the periodicity condition we have

$$(4.24) \quad w_n(z) = w_n(z', z_d) = \sum_{\xi' \in \mathbb{Z}^{d-1}} c_{\xi'}(n; z_d) e^{2\pi i \xi' \cdot z'} = \sum_{\xi \in \mathbb{Z}^{d-1} \times \{0\}} c_\xi(n; z_d) e^{2\pi i \xi \cdot z},$$

where  $(z', z_d) \in \mathbb{R}^{d-1} \times [0, \infty)$  and for  $\xi = (\xi', 0) \in \mathbb{Z}^{d-1} \times \{0\}$  we let

$$(4.25) \quad c_\xi(n; z_d) = \int_{\mathbb{T}^{d-1}} w_n(z', z_d) e^{-2\pi i \xi \cdot z} dz'$$

be the  $\xi$ -th Fourier coefficient of  $w_n(\cdot, z_d)$ . By the construction of  $w_n$  for any  $\xi \in \mathbb{Z}^{d-1} \times \{0\}$  we have

$$(4.26) \quad c_\xi(n; 0) = c_\xi(v_0), \quad \forall n \in \mathbb{S}^{d-1} \text{ satisfying } n_d > 0,$$

where  $c_\xi(v_0)$  is the corresponding Fourier coefficient of the fixed boundary data  $v_0$  involved in (4.12). The definition of  $T_n$  yields

$$z_d = T_n^{-1} y \cdot e_d = y \cdot (T_n^{-1})^t e_d = \frac{y \cdot n}{n_d},$$

and  $z' = y'$ . Using these relations between  $y$  and  $z$ , from (4.24) for the solution of  $v_n$  of (4.12) we obtain

$$(4.27) \quad v_n(y) = w_n(T_n^{-1} y) = \sum_{\xi \in \mathbb{Z}^{d-1} \times \{0\}} c_\xi \left( n; \frac{y \cdot n}{n_d} \right) e^{2\pi i \xi \cdot y}, \quad y \in \overline{\Omega_n}.$$

Observe that in view of the smoothness of  $w_n$  the function  $t \mapsto c_\xi(n; t) \in \mathbb{R}^N$  is smooth on  $[0, \infty)$  for each  $\xi \in \mathbb{Z}^{d-1} \times \{0\}$ . What we show next is a stability result with respect to normal vector  $n$  for the derivative of this function.

**Lemma 4.8.** *Fix  $\delta > 0$  small, and let  $\nu, \mu \in \mathbb{S}^{d-1}$  satisfy  $\nu_d, \mu_d \geq \delta$ . Then there exists a constant  $C_\delta = C(\delta, A)$  such that for any  $t \geq 0$  and all  $\xi \in \mathbb{Z}^{d-1} \times \{0\}$  one has*

$$|\partial_t c_\xi(\nu; t) - \partial_t c_\xi(\mu; t)| \leq C_\delta |\nu - \mu| \times \|v_0\|_{C^2(\mathbb{T}^d)},$$

where  $c_\xi(n; t)$  is given by (4.25).

**Proof.** It is clear from (4.25) that it suffices to prove stability of  $\nabla w_n$  with respect to  $n$ . Recall the notation  $G = \mathbb{T}^{d-1} \times \mathbb{R}_+$ , and set  $u = w_\nu - w_\mu$ . We get that  $u$  is a smooth solution to

$$(4.28) \quad \begin{cases} -\nabla \cdot A_\nu(z) \nabla u(z) = \nabla \cdot F(z), & z \in \mathbb{R}_+^d, \\ u(z', 0) = 0, & z' \in \mathbb{R}^{d-1}, \end{cases}$$

where we have denoted  $F(z) := (A_\nu(z) - A_\mu(z)) \nabla w_\mu(z)$ . Observe that  $u(\cdot, z_d)$ , as well as  $F(\cdot, z_d)$  are periodic with respect to  $\mathbb{Z}^{d-1}$  for any  $z_d \geq 0$ . Also, due to the construction it follows that  $u \in V_\tau(G)$  for some  $\tau > 0$  which will be specified in a moment. Since solution to (4.28) is unique in the space  $V_\tau$ , we may apply estimate (4.9) of Theorem 4.2 and by so obtain

$$(4.29) \quad \|u\|_{V_\tau(G; \mathbb{R}^{d \times N})} \leq \frac{C \|e^{\tau z_d} F(z)\|_{L^2(G; \mathbb{R}^{d \times N})}}{\tau \lambda_{A_\nu} - 2\tau \|A_\nu\|_{L^\infty(\mathbb{R}^d)}},$$

where in (4.29) we are following notation of Theorem 4.2. Using (4.2) and (4.14) from the definition of  $A_\nu$  we have  $\|A_\nu\|_{L^\infty(\mathbb{R}^d)} \leq C \|A\|_{L^\infty(\mathbb{T}^d)} \delta^{-2}$ . The latter combined with (4.19) implies that  $\tau = \frac{\lambda_A}{4\|A\|_\infty} \delta^4$  is a valid choice in (4.29), where  $\lambda_A$  is the ellipticity constant of the original operator in (4.12). Thus we will keep in mind that we have a uniform control over  $\tau$  in terms of the threshold  $\delta$ . Next, by (4.15) and (4.14) we easily get

$$(4.30) \quad \|A_\nu(z) - A_\mu(z)\|_{L^\infty(\mathbb{R}^d)} \leq C_\delta |\nu - \mu|,$$

which in combination with the choice of  $\tau$  and (4.29) infers

$$(4.31) \quad \|u\|_{V_\tau(G)} \leq C_\delta |\nu - \mu| \times \|w_\mu\|_{V_\tau(G)} \leq C_\delta |\nu - \mu| \times \|v_0\|_{C^1(\mathbb{T}^d)},$$

where the second inequality in (4.31) is due to (4.11).

Now fix some  $z_0 \in \partial \mathbb{R}_+^d$ , and for  $r > 0$  denote by  $\mathcal{K}(z_0, r)$  the intersection of a cube with side length  $r$  and center at  $z_0$  with  $\mathbb{R}_+^d$ . By boundary Schauder estimates (see [9] Theorem 5.21 and its proof) we have

$$(4.32) \quad \|\nabla u\|_{C^{0,\sigma}(\mathcal{K}(z_0, 1/2))} \lesssim_\delta \|\nabla u\|_{L^2(\mathcal{K}(z_0, 1))} + \|F\|_{C^{0,\sigma}(\mathcal{K}(z_0, 1))},$$

where  $0 < \sigma < 1$  is any fixed parameter, and the dependence of the constant in the inequality on parameter  $\delta$  comes from the dependence of the ellipticity constant of  $A_\nu$  on  $\delta$ . It is clear that

$$(4.33) \quad \|\nabla u\|_{L^2(\mathcal{K}(z_0, 1))} \leq \|u\|_{V_\tau(G)}.$$

Next, using the definition of  $F$  we have

$$(4.34) \quad \|F\|_{C^{0,\sigma}(\mathcal{K}(z_0, 1))} \leq \|A_\nu(z) - A_\mu(z)\|_{C^{0,\sigma}(\mathcal{K}(z_0, 1))} \|\nabla w_\mu\|_{C^{0,\sigma}(\mathcal{K}(z_0, 1))}.$$

The first factor in the right-hand side of (4.34) is easily seen, as in (4.30), to be bounded by  $C_\delta |\nu - \mu|$ . For the second one, we do a recourse to the construction of  $w_\mu$  in Corollary 4.3 and again using Schauder estimates at the boundary we get

$$\|\nabla w_\mu\|_{C^{0,\sigma}(\mathcal{K}(z_0, 1))} \lesssim_\delta \|\nabla w_\mu\|_{L^2(\mathcal{K}(z_0, 2))} + \|A_\mu \nabla v_0\|_{C^{0,\sigma}(\mathcal{K}(z_0, 2))} + \|v_0\|_{C^2(\mathcal{K}(z_0, 2))}.$$

In the last expression we estimate the  $L^2$ -norm of the gradient of  $w_\mu$  by  $V_\tau$  norm, which, on its turn, is controlled by (4.11). Getting back to (4.34) we obtain

$$(4.35) \quad \|F\|_{C^{0,\sigma}(\mathcal{K}(z_0, 1))} \leq C_\delta \|v_0\|_{C^2(\mathbb{T}^d)} |\nu - \mu|.$$

We now use (4.35), (4.33) and (4.31) in (4.32) to get

$$(4.36) \quad \|\nabla u\|_{L^\infty(\mathcal{K}(z_0, 1/2))} \leq C_\delta \|v_0\|_{C^2(\mathbb{T}^d)} |\nu - \mu|.$$

The claim of the lemma now follows directly by taking the derivative under the integral sign in (4.25) and applying (4.36). The proof is complete.  $\square$



**4.4. Boundary layer correctors.** For irrational direction  $n \in \mathbb{S}^{d-1}$  satisfying  $n \cdot \nu_0 > 0$ , and for fixed  $1 \leq \gamma \leq d$  let  $v_n^{*,\gamma}$  be the solution to (2.3) in a sense of Theorem 2.1. Under assumption (A5) on the operator we apply  $\nu_0 \cdot \nabla$  on both sides of the system in (2.4) and get that  $\chi^{*,\gamma}$ , the solution to the cell-problem, is also independent of  $\nu_0$ . We next fix a  $d \times d$ -matrix  $T_0$  with integer entries such that  $T_0 e_d = \nu_0$  and<sup>6</sup>  $\det T_0 \neq 0$ . Making a change of variables in (2.3) by setting  $y = T_0 z$ , and observing that  $y \cdot n = z \cdot T_0^t n$ , we transform the problem for boundary layer corrector to

$$(4.37) \quad \begin{cases} -\nabla_z \cdot \tilde{A}(T_0 z) \nabla_z \tilde{v}_n^\gamma(z) = 0, & z \in \Omega_{T_0^t n}, \\ \tilde{v}_n^\gamma(z) = -\tilde{\chi}^\gamma(z), & z \in \partial\Omega_{T_0^t n}, \end{cases}$$

where we have set  $v_n^{*,\gamma}(T_0 z) = \tilde{v}_n^\gamma(z)$ ,  $\chi^{*,\gamma}(T_0 z) = \tilde{\chi}^\gamma(z)$ , and the coefficients are being transformed as in Section 4.1. By the formula (4.1) we have

$$\nu_0 \cdot \nabla_y = T_0 e_d \cdot (T_0^{-1})^t \nabla_z = e_d \cdot \nabla_z,$$

hence both the operator and the boundary data in (4.37) are independent of  $e_d$ . Moreover, as  $T_0$  has integer entries, it follows that coefficients of (4.37) as well as the boundary data are periodic with respect to  $\mathbb{Z}^d$ . It is also clear that by the irrationality of  $n$  and the choice of  $T_0$  we have  $T_0^t n \notin \mathbb{R}\mathbb{Q}^d$ . Also,  $\tilde{v}_n^\gamma(z)$  is the solution of (4.37) in a sense of Theorem 2.1 if and only if  $\tilde{v}_n^\gamma(T_0 z)$  is the solution to (4.38) in a sense of Theorem 2.1. Finally noticing that  $T_0^t n \cdot e_d = n \cdot \nu_0 > 0$ , in (4.37) we are now in a position to apply the analysis of Section 4.3. In particular, from (4.27) we get that  $\tilde{v}_n^\gamma$ , the solution to (4.37), has the following expansion

$$\tilde{v}_n^\gamma(z) = \sum_{\xi \in \mathbb{Z}^{d-1} \times \{0\}} c_\xi^\gamma \left( T_0^t n; \frac{z \cdot T_0^t n}{T_0^t n \cdot e_d} \right) e^{2\pi i \xi \cdot z}, \quad z \in \overline{\Omega_{T_0^t n}},$$

where Fourier coefficients  $c_\xi^\gamma$  are defined as in (4.25). Since  $T_0^{-1} y \cdot T_0^t n = y \cdot n$  and  $T_0^t \cdot e_d = n \cdot \nu_0$  we finally get

$$(4.38) \quad v_n^{*,\gamma}(y) = \sum_{\xi \in \mathbb{Z}^{d-1} \times \{0\}} c_\xi^\gamma \left( T_0^t n; \frac{y \cdot n}{n \cdot \nu_0} \right) e^{2\pi i (T_0^{-1})^t \xi \cdot y}, \quad y \in \overline{\Omega_n},$$

for the solution of (2.3).

Clearly, the entire analysis remains valid for irrational directions  $n$  satisfying  $n \cdot \nu_0 < 0$ .

**Proof of Theorem 1.3.** For  $\tau > 0$  set  $S_{\tau,+} = \{n \in \mathbb{S}^{d-1} : n \notin \mathbb{R}\mathbb{Q}^d, n \cdot \nu_0 > \tau\}$ . For  $\xi \in \mathbb{Z}^d$  consider the function  $v_\xi(y) := e^{2\pi i \xi \cdot y} I_N$ ,  $y \in \mathbb{R}^d$ , where  $I_N \in M_N(\mathbb{R})$  is the identity matrix.

Let  $n \in S_{\tau,+}$ , and consider a boundary layer system (2.1) set on  $\Omega_n$  and with boundary data  $v_\xi$ . Let  $v_\xi^\infty \in M_N(\mathbb{R})$  be the corresponding constant field given by Theorem 2.1. The formula (2.5) for  $v_\xi^\infty$  in view of (2.7) and (2.8) is reduced to

$$(4.39) \quad v_\xi^\infty(n) = \int_{\partial\Omega_n} G^0(n, y) d\sigma(y) \times [c_\xi(A^{\beta\alpha}) n_\beta + c_\xi(\partial_{y_\beta}(\chi^{*,\alpha})^t A^{\beta\gamma}) n_\gamma + \mathcal{M}\{\partial_{y_\beta}(v_n^{*,\alpha})^t e^{2\pi i \xi \cdot y} A^{\beta\gamma}\} n_\gamma],$$

<sup>6</sup>For our arguments it is enough to have existence of the inverse of  $T_0$  with rational entries, however, it is useful to observe that with a little extra work one may assure  $\det T_0 = 1$  provided the greatest common divisor of the components of  $\nu_0$  equals one. The latter can always be assumed without loss of generality, as the condition (A5) is invariant under scaling of  $\nu_0$ . We will show this fact in Claim 4.9. The advantage of having  $\det T_0 = 1$  lies in the fact that the inverse of  $T_0$  will also have integer entries, which ensures that all boundary layer correctors  $v_n^{*,\gamma}$  defined in (4.38) remain periodic with respect to  $\mathbb{Z}^{d-1}$  in tangential directions.

where  $c_\xi$  is the  $\xi$ -th Fourier coefficient and  $v_n^{*,\alpha}$  is the solution to (2.3). We are going to apply Corollary 2.5 to the last term in (4.39), and as a set of parameters  $\mathcal{E}$  we take  $S_{\tau,+}$ . Observe that for  $v_n^{*,\alpha}$  we have the expansion (4.38). Hence, by (4.26), and Lemma 4.8 combined with the bound  $\|v_\xi\|_{C^2(\mathbb{T}^d)} \lesssim |\xi|^2$ , we have that the family of functions  $\{\partial_{y_\beta} v_n^{*,\alpha}\}_{n \in S_{\tau,+}}$  satisfies all requirements of Corollary 2.5. Next, applying the corollary with the smooth function  $e^{2\pi i \xi \cdot y} A^{\beta\gamma}$  we get that the mapping  $n \mapsto \mathcal{M}\{\partial_{y_\beta} (v_n^{*,\alpha})^t e^{2\pi i \xi \cdot y} A^{\beta\gamma}\}$  is Lipschitz continuous on  $S_{\tau,+}$  with Lipschitz constant bounded by  $C_\tau \sum_{\eta \in \mathbb{Z}^d} |c_\eta(A^{\beta\gamma})| |\eta|^2$ , where  $C_\tau$  is independent of  $n$ . Finally, combining this with Lemma 3.3, from (4.39) we get

$$(4.40) \quad |v_\xi^\infty(n) - v_\xi^\infty(\nu)| \leq C_\tau |n - \nu|, \quad n, \nu \in S_{\tau,+},$$

where  $C_\tau$  is independent of  $\xi$ .

For  $\tau > 0$  define  $D_{\tau,+} = \{x \in \partial D : n(x) \notin \mathbb{RQ}^d, n(x) \cdot \nu_0 > \tau\}$ , where  $n(x)$  is the normal inward vector of  $\partial D$  at  $x$ . Following the notation of Section 2, for any  $x, y \in \partial D_{\tau,+}$  by (2.2) we have

$$(4.41) \quad |g^*(x) - g^*(y)| \leq \sum_{\xi \in \mathbb{Z}^d} |g_\xi^*(x) - g_\xi^*(y)| \leq \sum_{\xi \in \mathbb{Z}^d} |c_\xi(x)| \times |v_\xi^\infty(n(x)) - v_\xi^\infty(n(y))| + \sum_{\xi \in \mathbb{Z}^d} |v_\xi^\infty(n(x))| \times |c_\xi(x) - c_\xi(y)| =: \Sigma_1 + \Sigma_2.$$

Recall that  $c_\xi(x) = \int_{\mathbb{T}^d} g(x, z) e^{-2\pi i \xi \cdot z} dz$ , where  $\xi \in \mathbb{Z}^d$ ,  $x \in \partial D$ . Fix a non zero  $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{Z}^d$  and let  $|\xi_\alpha| = \max_{1 \leq \beta \leq d} |\xi_\beta|$ . Let also  $\partial_{2,\alpha}^{d+1}$  be the partial differentiation operator acting on  $g(x, \cdot)$  ( $d+1$ )-times in the  $\alpha$ -th coordinate. Using the smoothness of  $g$ , from the definition of  $c_\xi(x)$  we get

$$c_\xi(x) = \frac{1}{(-2\pi i \xi_\alpha)^{d+1}} \int_{\mathbb{T}^d} \partial_{2,\alpha}^{d+1} g(x, z) e^{-2\pi i \xi \cdot z} dz.$$

Combining this with a uniform bound on  $|c_0(\cdot)|$  we get

$$(4.42) \quad |c_\xi(x)| \lesssim_g (1 + |\xi|)^{-(d+1)} \text{ uniformly in } x \in \partial D \text{ and } \xi \in \mathbb{Z}^d.$$

As  $g$  is smooth with respect to both of its variables, in a similar way we obtain

$$(4.43) \quad |c_\xi(x) - c_\xi(y)| \lesssim_g (1 + |\xi|)^{-(d+1)} |x - y|,$$

for all  $x, y \in \partial D$  and non zero  $\xi \in \mathbb{Z}^d$ . Using (4.42) and (4.40) for any  $x, y \in \partial D_{\tau,+}$  we get  $\Sigma_1 \leq C_{g,\tau} |n(x) - n(y)| \leq C_{g,\tau} |x - y|$ , where we have used the smoothness of  $\partial D$  to obtain the second inequality. In a similar vein, in  $\Sigma_2$  using Lemma 2.6 to bound the constant field and employing (4.43) leads to  $\Sigma_2 \leq C_g |x - y|$ . The estimates for  $\Sigma_1$  and  $\Sigma_2$  applied to (4.41) show that  $g^*$  is Lipschitz on  $\partial D_{\tau,+}$ .

Obviously, the same argument works for the other hemisphere  $S_{\tau,-} = \{n \in \mathbb{S}^{d-1} : n \notin \mathbb{RQ}^d, n \cdot \nu_0 < -\tau\}$  as well. The proof of the Theorem is now completed.  $\square$

**4.5. Concluding remarks.** We finish the paper with a few observations and comments.

**Around the theme of regularity.** Results concerning regularity of boundary layer tails are very few in the literature. In the same setting as we have here, namely second order divergence type elliptic systems, the smoothness of  $g^*$  under condition (1.3) on the coefficients and in dimensions greater than two, was established by H. Shahgholian, P. Sjölin, and the current author in [2] as an outcome of methods of [2] and [12] (see formula (4.4) in [2], and the discussion after that). It is easy to see that we recover this result for  $g^*$  from the proof of Theorem 1.3 above. Namely, the condition (1.3) implies that solutions to

cell-problem (2.4), and hence to boundary layer systems (2.3), are trivial. This in its turn shows that in formula (4.39) the last average is vanishing, and we get that the boundary layer tail, as a function of normal  $n$ , equals to a  $C^\infty$  function almost everywhere on the sphere. The rest of the proof proceeds with minor modifications. In dimension two, the smoothness of  $g^*$  is new, while in dimensions larger than two we get an alternative proof of the mentioned result from [2].

Concerning other settings, the reader may consult a recent work by Feldman and Kim [7], and the references therein, where they analyse continuity properties of boundary layer tails associated with fully nonlinear uniformly elliptic equations of second order.

Getting back to our case, one can see from the analysis above that the main obstacle towards the regularity of  $g^*$  comes from boundary layer correctors, in particular we do not know if the behaviour of boundary layer tails  $v^\infty(n)$  is in any sense uniform with respect to normals  $n$ . A specific instance of this non uniformity is the convergence speed of boundary layer correctors to their corresponding tails away from the boundary. Concerning this aspect in [1] we show that given any one-to-one, continuous function decreasing to 0 at infinity (i.e. a convergence rate), one may construct a problem of form (2.1) with smooth data, so that convergence towards boundary layer tail is slower than the given rate in advance. This in particular indicates that approaches towards regularity of  $g^*$  which are based on controlling the speed of convergence of the tails, are unlikely to lead to a positive conclusion.

It is also interesting to observe (in the light of Example 4.1) that the condition (A5) implies that the operator only “sees” Diophantine directions on the hemispheres considered in the proof of Theorem 1.3. It thus leads to an idea that one may try to tailor the Diophantine condition of [8] to the given operator. Developing this line it seems plausible that one should be able to deduce the claim of Theorem 1.2 (although without any structural results such as expansion (4.38)) using instead methods of [8] combined with some of the ideas considered here, in particular Lemma 3.4 and the proof of Theorem 1.3. In this perspective the approach of Section 4 should be seen as a more transparent alternative to some of the methods of [8] under condition (A5), and it will be interesting to see if the ideas considered here can be developed to lead to actual homogenization of the problem (1.1)-(1.2) under conditions (A1)-(A5). Finally, we note that we do not know if the regularity we have for  $g^*$  is in some sense sharp. Also, it will be very interesting to see if the regularity of  $g^*$  can have some impact on the speed of convergence in the actual homogenization problem (1.1)-(1.2). A positive sign in this direction is Theorem 1.2, although there the smoothness of  $g^*$  is a corollary, rather than a starting point.

**An element of  $\mathrm{SL}(d, \mathbb{Z})$  with prescribed column.** We finish the paper by showing that the integer matrix  $T_0$  which was used to transform  $e_d$  to the given vector  $\nu_0$  in subsection 4.4 can be chosen so that its inverse also has integer elements. This fact can be used to get periodicity of boundary layer correctors in tangential directions. For given integers  $(a_1, \dots, a_d) \in \mathbb{Z}^d$  we denote by  $[a_1, \dots, a_d]$  their greatest common divisor. We also recall a standard notation  $\mathrm{SL}(d, \mathbb{Z})$  for the *special linear group* over integers.

**Claim 4.9.** *For any  $a = (a_1, \dots, a_d) \in \mathbb{Z}^d$  such that  $[a_1, \dots, a_d] = 1$  there exists  $T \in \mathrm{SL}(d, \mathbb{Z})$  satisfying  $Te_d = a$ .*

**Proof.** Before we start observe that the condition on greatest common divisor to be one is necessary which trivially follows from Euclid’s algorithm.

The proof is by induction on  $d$ . When  $d = 2$  the existence of  $T$  follows directly from Euclid’s algorithm. We now show how to construct  $T$  for  $d = 3$  as the general case is similar. Let now  $d = 3$ , then  $[a_1, [a_2, a_3]] = 1$  and by the two-dimensional case there exist

$s_{11}, s_{21} \in \mathbb{Z}$  such that  $\begin{pmatrix} s_{11} & a_1 \\ s_{21} & [a_2, a_3] \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ . In view of the Euclid's algorithm there exist  $x, y \in \mathbb{Z}$  satisfying  $[a_2, a_3] = xa_2 + ya_3$ . Observe that necessarily we have  $[x, y] = 1$ . Now consider the matrix  $\begin{pmatrix} s_{11} & 0 & a_1 \\ *_1 & y & a_2 \\ *_2 & -x & a_3 \end{pmatrix}$  where we need to choose  $*_1, *_2 \in \mathbb{Z}$  so that it has determinant one. Expansion by the first row shows that it is enough to have the cofactor of  $a_1$  equal to  $s_{21}$ . But this is possible since  $x$  and  $y$  being coprime implies  $\mathbb{Z} = x\mathbb{Z} + y\mathbb{Z}$  by Euclid's algorithm, and hence the case of  $d = 3$  is proved.

The case of general  $d$  proceeds by constructing the matrix for  $(a_1, \dots, a_{d-1}, [a_d, a_{d+1}])$ , such that all elements above the main diagonal are zero except those on the last column. We then add another row to the matrix by choosing  $x, y \in \mathbb{Z}$  and writing  $[a_d, a_{d+1}] = \det \begin{pmatrix} x & a_d \\ y & a_{d+1} \end{pmatrix}$ . Then using the fact that  $[x, y] = 1$  we successively construct the last two rows of the new  $(d + 1)$ -dimensional matrix so that its determinant when expanded by the first row coincides with the one constructed for  $d$  elements, and hence will be equal to 1. The details are easy to fill, and so the proof is complete.  $\square$

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#### REFERENCES

- [1] Aleksanyan, H.: Slow convergence in periodic homogenization problems for divergence type elliptic operators. (in preparation)
- [2] Aleksanyan, H., Shahgholian, H., Sjölin, P.: Applications of Fourier analysis in homogenization of Dirichlet problem. *L<sup>p</sup> estimates. Arch. Ration. Mech. Anal. (ARMA)* **215**(1), 65-87 (2015)
- [3] Avellaneda, M., Lin, F.: Compactness methods in the theory of homogenization. *Commun. Pure Appl. Math.*, **40**(6), 803-847 (1987)
- [4] Bensoussan, A., Lions, J.-L., Papanicolaou, G.: *Asymptotic Analysis For Periodic Structures*. AMS (2011)
- [5] Bott, R., Milnor, J.: On the parallelizability of spheres, *Bull. AMS* **64**, 87-89 (1958)
- [6] Dong, H., Kim, S.: Green's matrices for second order elliptic systems with measurable coefficients in two dimensional domains. *Trans. Amer. Math. Soc.*, **361**, 3303-3323 (2009)
- [7] Feldman, W., Kim, I.: Continuity and Discontinuity of the Boundary Layer Tail. arXiv:1502.00966 (2015)
- [8] Gérard-Varet, D., Masmoudi, N.: Homogenization and boundary layers. *Acta Math.* 209, 133-178 (2012)
- [9] Giaquinta, M., Martinazzi, L.: *An introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs*. Scuola Normale Superiore Pisa (Lecture Notes) (2012)
- [10] Y.P., Hong, C.-T., Pan: A lower bound for the smallest singular value. *Linear Algebra and its Applications* **172**, 27-32 (1992)
- [11] Hofmann, S.; Kim, S.: The Green function estimates for strongly elliptic systems of second order. *Manuscripta Math.* (**124**), 139-172 (2007)
- [12] Kenig, C. E., Lin, F., Shen, Z.: Periodic Homogenization of Green and Neumann Functions. *Commun. Pure Appl. Math.* **67**(8), 1219-1262 (2014)
- [13] Kenig, C., Shen, Z.: Layer potential methods for elliptic homogenization problems. *Commun. Pure Appl. Math.* **64**(1), 1-44 (2011)
- [14] Kervaire, M., Milnor, J.: Groups of homotopy spheres: I. *Annals of Math.* **77**(3) 504-537 (1963)
- [15] Lions, J.L.: *Some methods in mathematical analysis of systems and their control*. Science Press, Beijing, Gordon and Breach, New York (1981)

- [16] Neuss-Radu, M.: The boundary behavior of a composite material. *Mathematical Modelling and Numerical Analysis* **35**(3), 407-435 (2001)
- [17] Prange, C.: Asymptotic analysis of boundary layer correctors in periodic homogenization. *SIAM J. Math. Anal.*, **45**(1), 345-387 (2012)
- [18] Tartar, L.: *The general theory of homogenization: a personalized introduction* Vol. 7. Springer (2009)
- [19] Schulze, B.-W., Wildenhain G.: *Methoden der Potentialtheorie für elliptische Differentialgleichungen beliebiger Ordnung*, Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften: Mathematische Reihe 60, Birkhäuser-Verlag, Basel (1977)
- [20] Stein, E.: *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press (1993)
- [21] Šubin, M.: Differential and psedudifferential operators in spaces of almost periodic functions, *Math. Sb. (N.S.)*, 95(137), 560-587 (1974).

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